2.3 Integration of Complex Functions

So far we have only integrated nonnegative functions. Now we will extend to general extended real-valued or complex-valued functions.

Recall that if $f: X \to \overline{\mathbb{R}}$, then the functions

$$f^+ = \max\{f, 0\} \quad \& \quad f^- = -\min\{f, 0\}$$

are m & nonnegative. Further,

$$f = f^+ - f^- \quad \& \quad |f| = f^+ + f^-.$$

Definition

If $f: X \to \overline{\mathbb{R}}$ is m and at least one of $\int f^+, \int f^-$ is finite, then we define

$$\int f = \int f^+ - \int f^-.$$

If both $\int f^+ = \infty$ & $\int f^- = \infty$, then $\int f$ is undefined.

Example

Even though the improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{b\to\infty}\int_0^b \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

exists, the Lebesgue integral of $\frac{\sin x}{x}$ on $[0,\infty)$ does not exist, because

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^+ dx = \infty = \int_0^\infty \left(\frac{\sin x}{x}\right)^- dx.$$

Notation

a. Given $f: X \to \overline{\mathbb{R}}$, if both

$$\int f^+ < \infty \quad \& \quad \int f^- < \infty$$

then we say that f is integrable.

b. Although this terminology is not used as commonly, it is convenient to say that f is an extended integrable function if

$$\text{EITHER} \int f^+ < \infty \quad \text{OR} \quad \int f^- < \infty.$$

That is, f is extended integrable $\iff \int f$ exists.

Exercise
show that

$$f \text{ is integrable} \iff \int |f| < \infty.$$

Definition
We let $L^1(X)$ denote the space of integrable functions on X:

$$L^1(X) = \left\{ f: X \to \bar{\mathbb{R}} : \int |f| < \infty \right\}$$

Note the implicit dependence on μ in this notation. There are many other common notations for this space, e.g.,

$$L^1,\ L^1(\mu),\ L^1(X,\mu),\ L^1(X,\mathcal{M},\mu),$$

$$L^1(d\mu),\ L^1(X,d\mu),\quad \text{etc.}$$

Definition
If $f: X \to \bar{\mathbb{R}}$ is integrable, then we define its <u>L^1-norm</u> to be

$$\|f\|_1 = \int |f| \, d\mu.$$

Again note the implicit dependence on μ in this notation.

Despite its name, the L^1 norm is not a norm.

Exercise

Show that $\|\cdot\|_1$ is a seminorm on $L^1(X)$, ie.,

a. $0 \le \|f\|_1 < \infty \quad \forall f \in L^1(X)$,

b. $\|cf\|_1 = |c|\,\|f\|_1 \quad \forall f \in L^1(X),\ \forall c \in \mathbb{R}$

c. Triangle Inequality:

$$\|f+g\|_1 \le \|f\|_1 + \|g\|_1 \quad \forall f, g \in L^1(X).$$

Show that, as a consequence, $L^1(X)$ is a vector space under the operations of function addition & scalar multiplication.

Remark

In order to be called a norm, $\|\cdot\|_1$ would have to have the additional property that

$$\|f\|_1 = 0 \implies f = 0.$$

Unfortunately, this is not true in general. Instead, show that we only have

$$\|f\|_1 = 0 \implies f = 0 \text{ a.e.}$$

The standard way to turn a seminorm into a norm is to form equivalence classes.

Exercise

a. Show that the relation

$$f \sim g \quad \text{if} \quad f = g \text{ a.e.}$$

is an equivalence relation on $L^1(X)$.

b. Let

$$\tilde{f} = \{g \in L^1(X) : f = g \text{ a.e.}\},$$

i.e., $\tilde{f}$ is the equivalence class of f under the relation $\sim$. Define

$$|||\tilde{f}|||_1 = \|f\|_1.$$

Show that this quantity is well-defined, i.e., it is independent of the choice of representative f.

c. Let $\widetilde{L^1(X)}$ be the quotient space with respect to this relation, i.e.,

$$\widetilde{L^1}(X) = L^1(X)/\sim \; = \{\tilde{f} : f \in L^1(X)\}.$$

Show that $\widetilde{L^1}(X)$ is a normed space with respect to $|||\cdot|||_1$. Note that the zero vector in $\widetilde{L^1}(X)$ is

$$\tilde{0} = \{g \in L^1(X) : g = 0 \text{ a.e.}\}$$

Remark

In effect, in passing from $L^1(X)$, whose elements are functions, to $\widetilde{L^1}(X)$, whose elements are equivalence classes of functions, we are "identifying" all functions that are equal a.e. In other words, if f & g are equal a.e., then they define the same equivalence class: $\tilde{f} = \tilde{g}$. Typically, we abuse notation and let the symbol "f" denote both the function f &

the equivalence class $\tilde{f}$ of all functions equal to f a.e., and we write $L^1(X)$ instead of $\widetilde{L^1}(X)$. We will do exactly this from now on, i.e., whenever we write $L^1(X)$ we really mean the space $\widetilde{L^1}(X)$. With this abuse of notation, any two functions that are equal a.e. "are" the same element of $L^1(X)$. In particular, any function that is zero a.e. "is" the zero vector in $L^1(X)$.

As long as we take a little care, ignoring the distinction between a function & its equivalence class is not usually much of a problem. One thing we have to careful of is that a "function" f in $L^1(X)$ does not have function values, as we can change the values on any set of measure zero and still have the *same* element of $L^1(X)$!

Examples

If we say "$f \in L^1(X)$ is continuous", we mean that f is equal a.e. to some continuous function, i.e., one of the representatives of the equivalence class of f is continuous.

On the other hand, if g is a continuous & integrable function, then, as an element of $L^1(X)$, g and any function equal to it a.e. determine the same element of $L^1(X)$.

Exercise

Show that $f \in L^1(X) \implies f$ is finite a.e.

Give an example showing that the converse need not hold.

Complex-Valued Functions

Definition

Given $f: X \to \mathbb{C}$ (m), write $f = \operatorname{Re} f + i \operatorname{Im} f$.

If $\int \operatorname{Re} f$ and $\int \operatorname{Im} f$ both exist and are finite

then we define

$$\int f = \underbrace{\int \operatorname{Re} f + i \int \operatorname{Im} f}_{\text{Note this must be a complex number}}.$$

Note that $\int f$ exists if & only if both $\operatorname{Re} f, \operatorname{Im} f$ are integrable. In this case, we say f is integrable

Exercise

f is integrable $\iff \int |f| < \infty$.

Definition

$$L^1(X) = \left\{ f: X \to \mathbb{C} \; : \; \int |f| < \infty \right\}$$

Note that we use the same symbols, $L^1(X)$, to denote the space of integrable extended-real functions and integrable complex functions. Which one is meant is usually clear from context.

All the same remarks apply: $L^1(X)$ is a seminormed space w.r.t. $\|f\|_1 = \int |f|$, and "becomes" a normed space when we identify functions that are equal a.e.

Exercise

Prove the linearity of the integral on $L^1(X)$, in both the extended real & complex settings:

$$\int (af+bg) = a\int f + b\int g \quad \forall f,g \in L^1(X), \ \forall \text{ scalars } a,b$$

Scalars are either real or complex according to context.

Basic Properties of the Integral

Theorem

If $f \in L^1(X)$, then $\left|\int f\right| \le \int |f|$.

Proof:

Exercise: Do the real case.

For the complex case, Write

$$\left|\int f\right| = \alpha \int f \quad \text{where } |\alpha| = 1.$$

Then

$$\int \alpha f = \alpha \int f = \left|\int f\right| \quad \text{all are } \underline{\text{real}}.$$

Write

$$\alpha f = f_1 + i f_2 \quad \text{where } f_1, f_2 \text{ are real.}$$

Then

$$\left|\int f\right| = \int \alpha f = \int f_1 + i \int f_2 \quad \text{is } \underline{\text{real}}$$

so we must have $\int f_2 = 0$ (does not imply $f_2 = 0$ ae)

Therefore,

$$\left|\int f\right| = \int f_1 \underset{\text{real case}}{\le} \int |f_1| \underset{|f_1| \le |\alpha f| = |f|.}{\le} \int |f|$$

∎

Theorem

If $f, g \in L^1(X)$, then TFAE:

a. $\int_E f = \int_E g \quad \forall E \in \mathcal{M}$

b. $\|f-g\|_1 = \int |f-g| = 0.$

c. $f = g$ a.e.

Proof:

$a \Rightarrow c.$ Set

$$u = \mathrm{Re}(f-g), \qquad v = \mathrm{Im}(f-g).$$

Suppose that $E = \{u^+ \neq 0\}$ has positive measure.

Then since $u^-(x) = 0$ whenever $u^+(x) \neq 0$,

$$\int_E \mathrm{Re}(f-g) = \int_E u^+ - \cancelto{0}{\int_E u^-} > 0,$$

contradicting the fact that $\int_E (f-g) = 0.$

Thus $\{u^+ \neq 0\}$ has measure zero, & similarly so do

$\{u^- \neq 0\}$, $\{v^+ \neq 0\}$, and $\{v^- \neq 0\}$. Hence

$f - g = 0$ a.e. $\blacksquare$

$c \Rightarrow b$. If $f = g$ a.e. then $|f-g| = 0$ a.e., so

$\int |f-g| = 0$.

$b \Rightarrow a$. If $\int |f-g| = 0$ & $E \in \mathcal{M}$, then

$$0 \leq \left| \int_E f - \int_E g \right| \leq \int_E |f-g| \leq \int_X |f-g| = 0,$$

so $\int_E f = \int_E g$. ∎

Exercise: Reverse Triangle Inequality

$$\Big| \|f\|_1 - \|g\|_1 \Big| \leq \|f-g\|_1$$

(True for any norm.)

Definition: Convergence

If $f_n, f \in L^1(X)$, then we say that fn converges to f in L^1-norm if

$$\lim_{n\to\infty} \|f - f_n\|_1 = \lim_{n\to\infty} \int |f - f_n| = 0.$$

In this case, we write

$$f_n \longrightarrow f \quad \text{in } L^1(X).$$

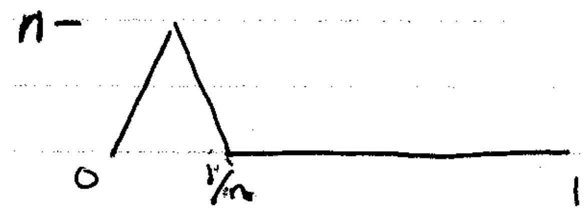
<u>Example</u>

a. Show that $\|f - f_n\|_1 \to 0 \implies \|f_n\|_1 \to \|f\|_1$

b. Show that $\|f_n\|_1 \to \|f\|_1 \not\Rightarrow \|f - f_n\|_1 \to 0$ in general

c. Show that $f_n \longrightarrow f$ pointwise $\not\Rightarrow f_n \to f$ in L^1.

Hint:

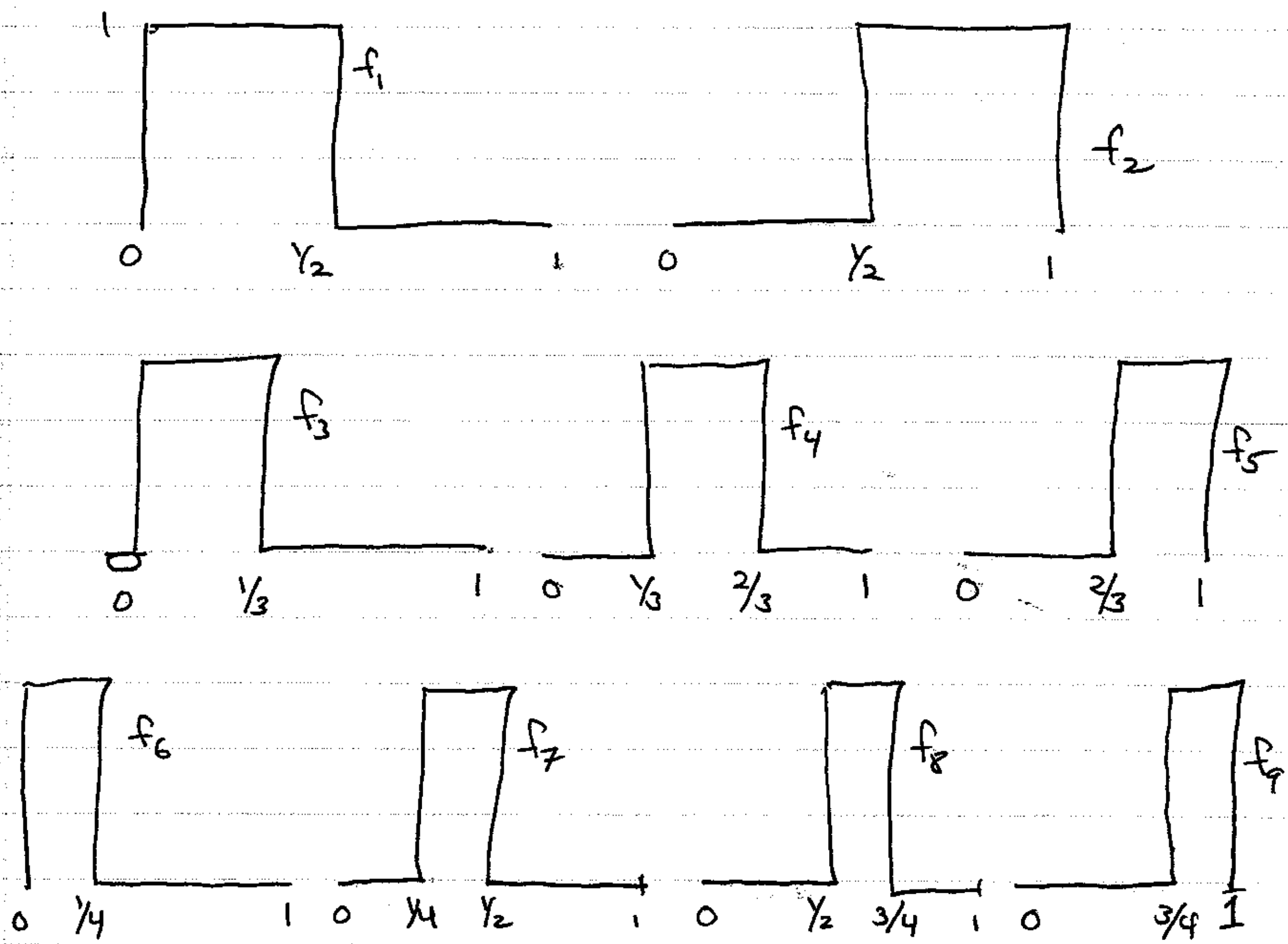


d. Show $f_n \longrightarrow f$ on $L^1 \not\Rightarrow f_n \to f$ pointwise.

Hint: Consider the "Marching Boxes" example

(see next page).

Marching Boxes



etc.

e. Find a <u>subsequence</u> $\{f_{n_k}\}_{k\in\mathbb{N}}$ of the marching boxes that <u>does</u> converge pointwise a.e.

<u>Remark</u>

We'll see later that

$f_n \to f$ in L^1 $\Longrightarrow$ $\exists$ subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges pointwise a.e. to f

Some additional remarks on $L^1(X)$

For a complete measure, we know that if f is Ⓜ & $g = f$ a.e., then g is Ⓜ. Thus if $f \in L^1(X)$, then every representative of the equivalence class of f is Ⓜ. What if μ is not complete? Fortunately, not much goes wrong if we replace μ by its completion $\bar{\mu}$ (defined in HW 2).

Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $(X, \overline{\mathcal{M}}, \bar{\mu})$ be its completion. If f is an extended real-valued or complex-valued $\overline{\mathcal{M}}$-measurable function on X, then $\exists$ an $\mathcal{M}$-measurable g on X s.t. $f = g$ $\bar{\mu}$-a.e.

Proof

Suppose that $A \in \overline{\mathcal{M}}$. Then, by definition,

$A = E \cup F$ where $E \in \mathcal{M}$ and $F \subseteq N$ for some μ-null set N. Hence $\chi_A = \chi_E$ $\bar{\mu}$-a.e., and χ_E is $\mathcal{M}$-measurable.

Now let f be an arbitrary $\overline{\mathcal{M}}$ - m function.

Exercise: Although we only did it for nonnegative functions, so $\exists$ simple functions ϕ_k that are $\overline{\mathcal{M}}$ - m and converge pointwise to f.

Since each ϕ_k is a ~~finear~~ linear combination of characteristic functions, write

$\phi_k = \sum_{j=1}^{N} a_j \chi_{A_j}$ with A_j $\overline{\mathcal{M}}$ - m. Then $\exists$ $\mathcal{M}$ - m E_j s.t. $\chi_{A_j} = \chi_{E_j}$ $\bar{\mu}$ - a.e. Hence, if we set $\psi_k = \sum_{j=1}^{N} a_j \chi_{E_j}$ then ψ_k is $\mathcal{M}$ - m and $\psi_k = \phi_k$ $\bar{\mu}$-a.e. In particular, if

$$E_k = \{\psi_k \neq \phi_k\}$$

Then $\bar{\mu}(E_k) = 0$.

Therefore $E = \bigcup E_k$ also satisfies

$\bar{\mu}(E) = 0$. By definition,

$$E = H \cup F$$

where $H \in \mathcal{M}$ and $F \subseteq N$ for some μ-null set N. Since $\mu(H) = \bar{\mu}(E) = 0$, we have that

$$M = H \cup N$$

is an $\mathcal{M}$-ⓜ null set that contains E. Set

$$f_k = \varphi_k \cdot \chi_{M^c}.$$

Each f_k is $\mathcal{M}$-ⓜ, so

$$g(x) = \lim_{k \to \infty} f_k(x) \quad \text{(exists } \forall x\text{, why?)}$$

is $\mathcal{M}$-ⓜ as well. Finally, $g = f$ $\bar{\mu}$-a.e. (why?)

∎

<u>Remark.</u>

In the converse direction, note that

f is $\mathcal{M}$-ⓜ $\Rightarrow$ f is $\overline{\mathcal{M}}$-ⓜ.

Convention

As a consequence of these results, whenever dealing with $L^1(X)$, we assume μ is complete (or replace μ by its completion).

Now we come to the "workhorse" of the convergence theorems!

Not the discrete cosine transform!!

Dominated Convergence Theorem (DCT)

Suppose that $f_n \in L^1(X)$ are such that

a. $f_n(x) \longrightarrow f(x)$ pointwise a.e.

b. $\exists g \in L^1(X)$ s.t. $|f_n| \le g$ a.e. $\forall n \in \mathbb{N}$

Then $f \in L^1(X)$ and

$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof:

If μ is complete, as in our convention, then f is (m) (and even if μ is not complete, if we redefine f on a null set then it will be Ⓜ).

Further,

$$|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq g(x) \text{ a.e.},$$

so

$$\int |f| \leq \int g < \infty.$$

Thus $f \in L^1(X)$.

Suppose that f is extended-real-valued. Then

$$g \pm f_n \geq 0 \text{ ae.}$$

Hence, by Fatou's Lemma,

$$\int g + \int f = \int (g+f)$$

$$= \int \liminf_{n\to\infty} (g+f_n)$$

$$\leq \liminf_{n\to\infty} \int (g+f_n) \qquad \text{Fatou.}$$

constant!

$$= \liminf_{n\to\infty} \left(\int g + \int f_n \right)$$

$$= \int g + \liminf_{n\to\infty} \int f_n .$$

Consequently,

$$\int f \le \liminf_{n\to\infty} \int f_n .$$

Similarly,

$$\int g - \int f = \int (g-f) \qquad \text{note all are } \underline{\text{finite}}$$

$$= \int \liminf_{n\to\infty} (g - f_n)$$

$$\le \liminf_{n\to\infty} \int (g - f_n) \qquad \text{Fatou}$$

$$= \liminf_{n\to\infty} \left(\int g - \int f_n \right)$$

$$= \int g + \liminf_{n\to\infty} \left(- \int f_n \right)$$

$$= \int g - \limsup_{n\to\infty} \int f_n$$

Hence

$$\limsup_{n\to\infty} \int f_n \le \int f.$$

Combining all the inequalities, we see that

$$\liminf_{n\to\infty} \int f_n = \int f = \limsup_{n\to\infty} \int f_n. \blacksquare$$

Exercise: An extension.

Show that with the same hypotheses we actually get the better conclusion that $f_n \to f$ in L^1-norm, i.e.,

$$\lim_{n\to\infty} \|f - f_n\|_1 = \lim_{n\to\infty} \int |f - f_n| = 0.$$

(refer to earlier exercises to see why this is better).

Hint: Apply the DCT to $|f - f_n|$.

Exercise: Another extension.

This is the Generalized DCT. Suppose $f_n \in L^1(X)$ and

a. $f_n(x) \longrightarrow f(x)$ pointwise a.e.

b. $\exists\, g_n \in L^1(X)$ s.t. $|f_n| \le g_n$ a.e.

c. $\exists\, g \in L^1(X)$ s.t. $g_n(x) \longrightarrow g(x)$ pointwise a.e.

Then $\lim_{n\to\infty} \int f_n = \int f$.

Hint: Closely follow the proof of the DCT.

As a corollary of the DCT, we get a result about switching integrals and infinite series. Recall that we had a similar theorem before, a special case of Tonelli's Theorem, but that only applied to nonnegative functions. The following result, which is a special case of Fubini's Theorem, applies to general functions.

Corollary

If $f_n \in L^1(X)$ and

$$\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$$

(in which case we say that $\sum f_n$ converges absolutely in L^1-norm)

then

a. $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for a.e. x,

b. $f \in L^1(X)$,

c. $\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$.

Proof:

Set
$$g_N(x) = \sum_{n=1}^{N} f_n(x) \quad \& \quad g(x) = \sum_{n=1}^{\infty} |f_n(x)|.$$

Converges in $[0,\infty]$

By Tonelli's Theorem, since all the terms are nonnegative,

$$\int g = \int \sum_{n=1}^{\infty} |f_n|$$
$$= \sum_{n=1}^{\infty} \int |f_n| \qquad \text{Tonelli}$$
$$= \sum_{n=1}^{\infty} \|f_n\|_1$$
$$< \infty$$

Therefore $g \in L^1(X)$ since $g \geq 0$.

Consequently, g must be finite a.e., i.e.,

$$g(x) = \sum_{n=1}^{\infty} |f_n(x)| < \infty \quad \text{except for } x \in Z$$
with $\mu(Z) = 0$.

Thus, for $x \notin Z$, as a series of scalars, the series

$$f(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} f_n(x) = \lim_{N\to\infty} g_N(x)$$

exists & is finite. That is, $g_N(x) \to f(x)$ pointwise a.e.

Since we also have

$$|g_N(x)| \le \sum_{n=1}^{N} |f_n(x)| \le \sum_{n=1}^{\infty} |f_n(x)|$$

$$= g(x) \in L^1(X)$$

we can apply the DCT to g_N to obtain that $f \in L^1(X)$ and

$$\int f = \lim_{N\to\infty} \int g_N$$

$$= \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \int f_n \qquad \text{linearity of the integral}$$

$$= \sum_{n=1}^{\infty} \int f_n \qquad \text{definition of infinite series.}$$

<u>Application</u>: Switching Derivatives and Integrals

Since a derivative is a limit, we might expect that, <u>with suitable hypotheses</u>, we can interchange an integral & a derivative (this is known as "differentiating under the integral sign").

The following result is typical. We will omit the proof, which can be found in Folland's text.

<u>Theorem</u>

Let $f: X \times [a,b] \longrightarrow \mathbb{C}$ be given. Suppose that

a. $f_t(x) = f(x,t) \in L^1(X) \quad \forall t \in [a,b]$

b. $\frac{\partial f}{\partial t}$ exists

c. $\exists g \in L^1(X)$ with $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq g(x) \quad \forall x, t$

Then

$$F(t) = \int f_t(x)\, d\mu(x), \qquad t \in [a,b]$$

is differentiable, and

$$F'(t) = \int \frac{\partial f}{\partial t}(x,t)\, d\mu(x).$$

Relation between Riemann & Lebesgue integrals

We sketched the relation before, assuming continuity of f. Let us give this a little more precisely now, without assuming continuity.

Definition (Darboux's definition of the Riemann integral).

Let $f: [a,b] \longrightarrow \mathbb{R}$ be bounded. Given a partition

$\Gamma = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$,

define

$$S_\Gamma f = \sum_{j=1}^{n} M_j (x_j - x_{j-1}) \qquad \text{upper Riemann sum}$$

$$s_\Gamma f = \sum_{j=1}^{n} m_j (x_j - x_{j-1}) \qquad \text{lower Riemann sum}$$

where

$$m_j = \inf_{[x_{j-1}, x_j]} f \quad \& \quad M_j = \sup_{[x_{j-1}, x_j]} f.$$

Set

$$\overline{I}_a^b(f) = \inf\{S_\Gamma f : \text{all partitions } \Gamma\}$$

$$\underline{I}_a^b(f) = \sup\{s_\Gamma f : \text{all partitions } \Gamma\}$$

Then the <u>Riemann Integral</u> of f exists if

$$\underline{I}_a^b(f) = \overline{I}_a^b(f),$$

and in this case we denote this number by

$$(R)\int_a^b f(x)\,dx$$

<u>Theorem</u>

Let $f:[a,b] \longrightarrow \mathbb{R}$ be bounded.

a. If f is Riemann integrable, then it is Lebesgue m and integrable, and

$$(R)\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dx.$$

b. f is Riemann integrable $\iff$ f is continuous a.e.

<u>Note</u>

Continuous a.e. means the set of discontinuities has measure zero. It does <u>not</u> mean that $\exists$ continuous function that equals f a.e.

Proof:

a. Assume f is Riemann integrable (and note we are not assuming f is continuous). Define simple functions

$$g_\Gamma = \sum_{j=1}^{n} m_j \chi_{[x_{j-1}, x_j]}, \quad G_\Gamma = \sum_{j=1}^{n} M_j \chi_{(x_{j-1}, x_j]}$$

Corresponding to a partition Γ. Then

$$s_\Gamma f = \int g_\Gamma \quad \& \quad S_\Gamma f = \int G_\Gamma \quad \text{(Lebesgue integrals!)}$$

Note that if Γ' refines Γ (includes all the points in Γ plus more), then

$$g_\Gamma \le g_{\Gamma'} \le G_{\Gamma'} \le G_\Gamma.$$

Now let

$$I = (R)\int_a^b f(x)\,dx = \sup\left\{\int g_\Gamma : \text{all partitions } \Gamma\right\}$$

Then there must exist a partition Γ_1 s.t.

$$I - 1 \le \int g_{\Gamma_1} \le I$$

Also, $\exists$ partition Γ_1' s.t.

$$I - \tfrac{1}{2} \le \int g_{\Gamma_1'} \le I$$

Set
$$\Gamma_2 = \Gamma_1 \cup \Gamma_1'.$$
Then Γ_2 is a refinement of both Γ_1 & Γ_1'. In particular,
$$g_{\Gamma_1'} \leq g_{\Gamma_2},$$
so
$$I - \tfrac{1}{2} \leq \int g_{\Gamma_1'} \leq \int g_{\Gamma_2} \leq I.$$

Continuing in this way, we can obtain refinements
$$\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$$
such that
$$g_{\Gamma_1} \leq g_{\Gamma_2} \leq \cdots$$
and
$$\int g_{\Gamma_k} \to I.$$
Let
$$g = \lim_{k \to \infty} g_{\Gamma_k}$$
This converges since the g_{Γ_k} are increasing, and

furthermore each $g_{\Gamma_k} \leq f$, which is a bounded function. Since we are on a finite domain & f is bounded, it is integrable. Also

$$|g_{\Gamma_k}| \leq |f| \in L^1[a,b] \quad \forall k.$$

Therefore, by the DCT,

$$\int g_{\Gamma_k} \longrightarrow \int g \text{ (Lebesgue integral)}$$

But also we know that $\int g_{\Gamma_k} \to I$,

so

$$\int g = I.$$

Note that, by construction, $g \leq f$, and g is m since it is a pointwise limit of the simple functions g_{Γ_k}.

Operating similarly from above, we can create a m function $G \geq f$ that satisfies

$$\int G = I.$$

Hence $G-g \geq 0$ & is ⓜ, and

$$\int (G-g) = \int G - \int g = I - I = 0.$$

Since $G-g$ is nonnegative, we conclude that

$G-g = 0$ a.e. And since $g \leq f \leq G$,

$$f = g = G \text{ a.e.}$$

Therefore f is ⓜ since Lebesgue measure is complete,

and

$$\underbrace{\int f}_{\substack{\text{Lebesgue}\\ \text{integral}}} = \int g = I = \mathcal{R}\int f.$$

b. $\Longrightarrow$ Suppose that f is Riemann integrable.

Using all the same notation as in part a, set

$$E = \{x \in X : f(x) = g(x) = G(x)\}.$$

Since $f = g = G$ a.e., $Z = E^C$ has measure zero.

Also, since each partition Γ_k contains only finitely many points,

$$\Gamma = \bigcup_k \Gamma_k$$

is a countable set.

Suppose that $x \notin \Gamma$ and f is discontinuous at x.

Exercise: Show $\exists\, \varepsilon > 0$ s.t.

$$G_{\Gamma_k}(x) - g_{\Gamma_k}(x) \geq \varepsilon \quad \forall\, k \in \mathbb{N}.$$

Letting $k \to \infty$, we see that

$$G(x) - g(x) \geq \varepsilon > 0,$$

so $x \notin E$, and therefore $x \in Z$. Hence f is continuous for all $x \notin \Gamma \cup Z$. This is a set of measure zero, so f is continuous a.e.

$\Leftarrow$ Suppose now that f is continuous a.e.

Choose <u>any</u> sequence of partitions Γ_k such that $|\Gamma_k| \to 0$. As in the proof of Problem #4 on HW 4,

$g_{\Gamma_k}(x),\ G_{\Gamma_k}(x) \longrightarrow f(x)$ at all points of continuity of f. By their definition, if we set

$$M = \sup_{x \in [a,b]} |f(x)|,$$

then

$$|g_{\Gamma_k}(x)|,\ |G_{\Gamma_k}(x)| \le M \quad \forall x \in [a,b].$$

Since $[a,b]$ is a finite interval, the constant function M is integrable, i.e., $M \in L^1[a,b]$. The DCT therefore implies that

$$s_{\Gamma_k} = \int g_{\Gamma_k} \longrightarrow \int f$$

$$S_{\Gamma_k} = \int G_{\Gamma_k} \longrightarrow \int f$$

Therefore

$$\underline{I}_a^b(f) = \sup_{\Gamma} \{s_\Gamma\} \ge \int f$$

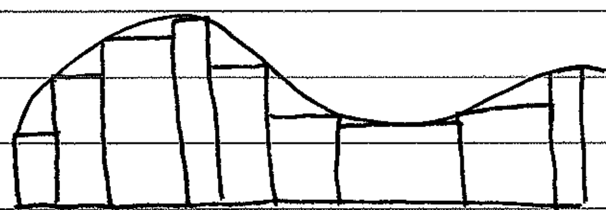
$$\overline{I}_a^b(b) = \inf_{\Gamma} \{S_\Gamma\} \le \int f$$

The opposite inequalities follow immediately from the definition, so f is Riemann integrable, &

$$\int f = \underline{I}_a^b(f) = \overline{I}_a^b(f) = (R)\int f. \blacksquare$$

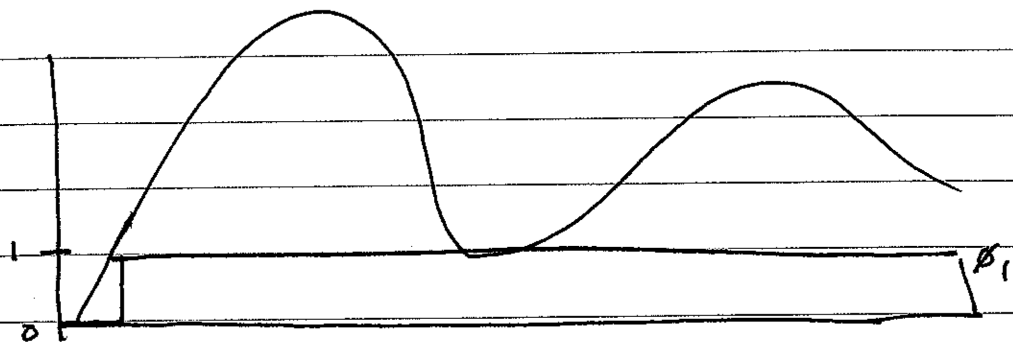
Comparison of Riemann & Lebesgue approaches

Riemann works by subdividing the x-axis:

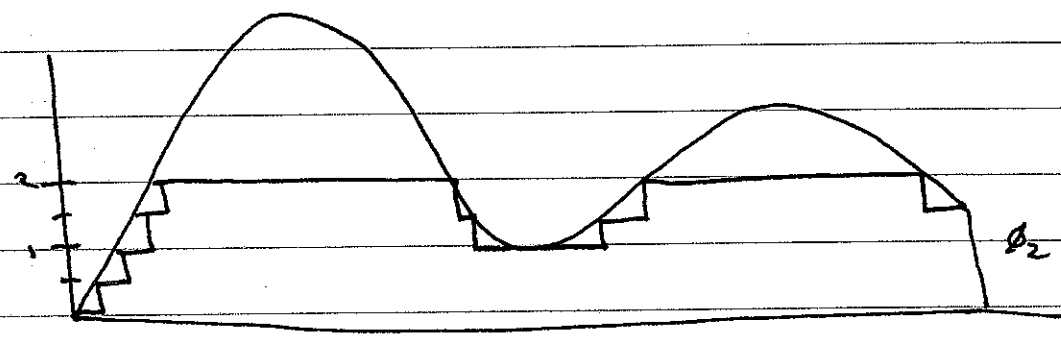


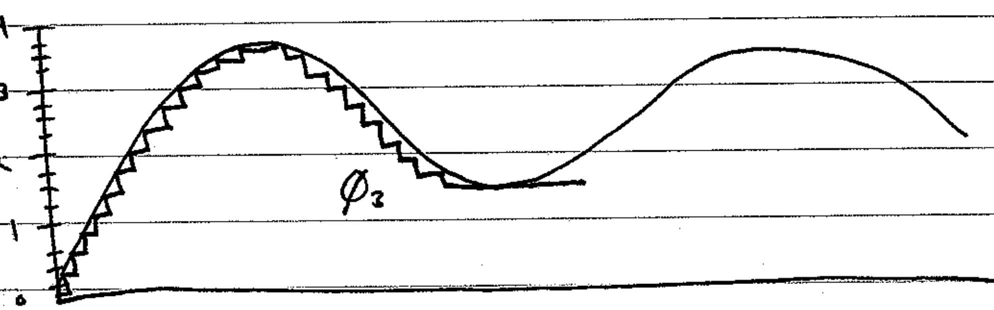
refine the partition

Lebesgue subdivides the y-axis (see picture next page — keep in mind that although it looks similar, that's because the illustration shows a continuous function).



Step 1





Disadvantages: More "complicated" def

Advantages: Wide class of fns are integrable,

Convergence Thms hold

Some Exercises

1. Prove the Bounded Convergence Theorem

Suppose μ is finite measure, and suppose

a. f_n, f are m functions on X (either extended-real or complex-valued),

b. $f_n(x) \longrightarrow f(x)$ a.e.

c. $\exists M$ s.t. $|f_n(x)| \le M$ a.e. $\forall n \in \mathbb{N}$

Then $\lim\limits_{n\to\infty} \int f_n = \int f$.

2. Prove the Uniform Convergence Theorem:

Let μ be a finite measure, and suppose

a. f_n, f are m functions on X (either extended-real or complex valued)

b. $f_n \longrightarrow f$ uniformly on X.

Then $\lim\limits_{n\to\infty} \int f_n = \int f$.

3. Suppose $f: \mathbb{R}^d \to \mathbb{R}$ is Lebesgue (m). Show that if $T: \mathbb{R}^d \to \mathbb{R}^d$ is a nonsingular linear transformation, then $f \circ T$ is (m).

Hint: Show $\{f \circ T > a\} = T^{-1}(\{f > a\})$.

Then show that any Lipschitz function (such as T^{-1}) maps (m) sets to (m) sets. (Recall we showed on Exam 1 that a Lipschitz function maps zero measure sets to zero measure sets. It is also true that a continuous function maps compact sets to compact sets. An arbitrary (m) function is the union of an F_σ set and a set of zero measure.)

4. Let $f_n, f: X \to [0, \infty]$ be (m), with $f_n \le f$ a.e. Show that if $f_n(x) \to f(x)$ for a.e. x, then $\int f_n \longrightarrow \int f$ (might be infinite).

5. Show that if $f_n \to f$ in $L^1(X)$, then $f_n \xrightarrow{m} f$ (see HW 4 for the definition of convergence in measure).

6. Let $f_n \in L^1(X)$ be given. Show that

$\sum_{k=1}^{\infty} f_k$ converges absolutely in $L^1(X)$, i.e., $\sum_{k=1}^{\infty} \|f_k\|_1 < \infty$,

implies $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely for a.e. x,

i.e., $\sum_{k=1}^{\infty} |f_k(x)| < \infty$ a.e.

Definition

The <u>essential supremum</u> of an extended real-valued function $f: X \to \overline{\mathbb{R}}$ is the smallest constant that dominates f almost everywhere:

$$\operatorname*{ess\,sup}_{x \in X} f(x) = \inf\{M : f \leq M \text{ a.e.}\}$$

The <u>L^∞-norm</u> of $f: X \to \overline{\mathbb{R}}$ or $f: X \to \mathbb{C}$ is

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in X} |f(x)|.$$

7. Show that if μ is a finite measure, then

$$\|f\|_1 \leq \mu(X) \cdot \|f\|_\infty$$

8. Show that $\nexists\, C > 0$ s.t.

$$\forall f \in L^1(\mathbb{R}), \quad \|f\|_1 \leq C\, \|f\|_\infty.$$

9. Show that if $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then

[redacted] $$\operatorname*{ess\,sup}_{x \in [a,b]} f(x) = \sup_{x \in [a,b]} f(x)$$

10. Let

$$L^\infty(X) = \left\{ f : X \underset{m}{\longrightarrow} \mathbb{C} \;:\; \|f\|_\infty < \infty \right\}$$

Show that $\|\cdot\|_\infty$ is a seminorm on $L^\infty(X)$, & is a norm if we identify functions that are equal a.e.

11. Show that if μ is a finite measure, then $L^\infty(X) \subseteq L^1(X)$.

Show that $L^\infty(\mathbb{R}) \not\subseteq L^1(\mathbb{R})$.