2.4 Modes of Convergence

There are many types of convergence criteria for functions. Here are a few, most of which we have seen before. Each type of convergence criterion quantifies one reasonable notion of what it means to "converge."

Pointwise convergence

$f_n \longrightarrow f$ pointwise if $\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in X$.

We can rewrite this in several ways, e.g.,

$$\lim_{n\to\infty} |f(x) - f_n(x)| = 0 \quad \forall x \in X$$

or

$$\forall x \in X \ \forall \varepsilon > 0 \ \exists N = N(x,\varepsilon) \text{ s.t. } n > N \Rightarrow |f(x) - f_n(x)| < \varepsilon$$

This is just convergence at each individual $x \in X$.

Pointwise a.e. convergence

Same as above, except we only require pointwise convergence at a.e. x: $f_n \to f$ pointwise a.e. if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for a.e. } x \in X.$$

Uniform Convergence

$f_n \longrightarrow f$ uniformly if

$$\lim_{n\to\infty} \left(\sup_{x\in X} |f(x) - f_n(x)| \right) = 0$$

Rewritten,

$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) \quad \text{s.t.} \quad \forall x \in X,$$

$$n > N \implies |f(x) - f_n(x)| \leq \varepsilon.$$

Note that N depends only ε only, and not on x.

Thus $f_n(x) \longrightarrow f(x)$ "at the same rate for each x".

Obviously,

$$\text{Uniform Convergence} \implies \text{Pointwise Convergence}$$

Exercise

a. Show that if $f_n \to f$ uniformly & each f_n is continuous, then f is continuous.

b. Show that $\exists$ continuous f_n that converge pointwise to a discontinuous function.

L^1-convergence

Whenever you have a norm, you have an associated notion of convergence. L^1 convergence is convergence w.r.t. the L^1-norm:

$$f_n \to f \text{ in } L^1(X) \quad \text{if} \quad \lim_{n\to\infty} \|f - f_n\|_1 = 0.$$

We can rewrite this as

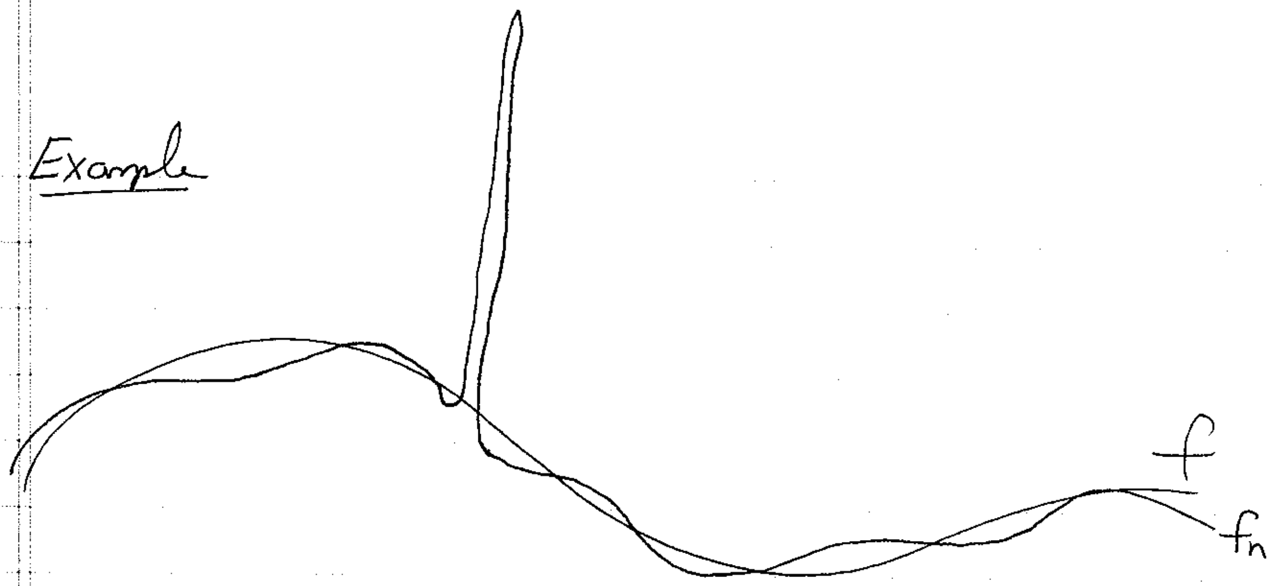
$$\lim_{n\to\infty} \int |f - f_n| = 0$$

or

$$\forall \varepsilon > 0,\ \exists N > 0 \text{ s.t. } n > N \Rightarrow \int |f - f_n| \leq \varepsilon.$$

Comparing to uniform convergence, we see that for L^1-convergence, the "distance" from f_n to f is determined by computing the <u>integral</u> of $|f - f_n|$, while for uniform convergence it is the <u>supremum</u> of $|f - f_n|$. Each is a completely appropriate notion of convergence — which is "better" will depend on your context.

Example



The supremum of $|f-f_n|$ will be large, but the integral of $|f-f_n|$ will be small.

L^∞-norm convergence

The L^∞-norm is like an "almost everywhere" supremum:

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in X} |f(x)|$$
$$= \inf\{M : |f(x)| \le M \text{ a.e.}\}$$

L^∞-convergence is convergence w.r.t. this norm:

$$f_n \to f \text{ in } L^\infty(X) \text{ if } \lim_{n\to\infty} \|f-f_n\|_\infty = 0.$$

We can ~~easily~~ rewrite this as

$$\lim_{n\to\infty} \left(\operatorname*{ess\,sup}_{x\in X} |f(x)-f_n(x)| \right) = 0.$$

Exercises

a. $f_n \to f$ in $L^\infty(X)$ $\Longrightarrow$ $f_n \to f$ pointwise a.e.

Give an example showing the converse can fail.

b. Show that if μ is finite, then

$$f_n \to f \text{ in } L^\infty(X) \implies f_n \to f \text{ in } L^1(X)$$

c. Show that the converse implication in part b can fail, even if μ is finite.

d. Show that part b can fail if we do not assume that μ is a finite measure.

Convergence in Measure

This is another useful quantification of what it means for functions to converge.

Definition

Let $f_n, f : X \to \mathbb{C}$ be ⓜ. Then we say that

<u>f_n converges in measure to f</u>, denoted $f_n \xrightarrow{m} f$, if

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mu\{|f - f_n| \geq \varepsilon\} = 0.$$

Remark

If μ is complete, then we can also apply the same definition to extended real-valued functions that are finite a.e. We need this restriction in order that $f - f_n$ will be ⓜ, even though it may be defined only almost everywhere.

The following is a simple, but extremely useful inequality.

Tchebyshev's Inequality

If $f \in L^1(X)$, then

$$\forall \varepsilon > 0, \quad \mu\{|f| \geq \varepsilon\} \leq \frac{1}{\varepsilon}\|f\|_1 = \frac{1}{\varepsilon}\int |f|.$$

Proof:

$$\begin{aligned} \|f\|_1 &= \int |f| \\ &\geq \int_{\{|f| \geq \varepsilon\}} |f| \\ &\geq \int_{\{|f| \geq \varepsilon\}} \varepsilon \\ &= \varepsilon\,\mu\{|f| \geq \varepsilon\}. \quad \blacksquare \end{aligned}$$

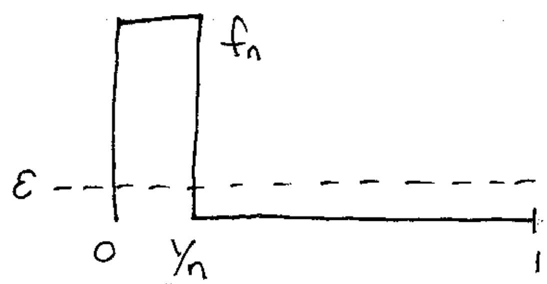
Exercise

Use Tchebyshev's Inequality to prove that

$$f_n \to f \text{ in } L^1(X) \implies f_n \xrightarrow{m} f.$$

One of our standard examples shows that the converse fails.

Example: Shrinking Boxes

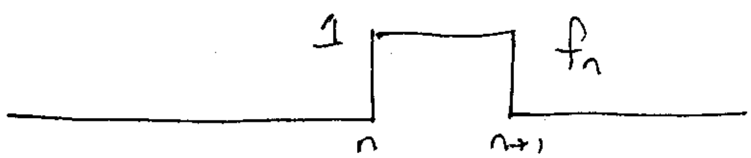


$\mu\{|f_n - 0| \geq \varepsilon\} = \frac{1}{n} \to 0$ (Lebesgue measure)

so $f_n \xrightarrow{m} 0$. However, $\|f_n - 0\|_1 = 1 \ \forall n$,

so $f_n \not\to 0$ in $L^1[0,1]$.

Exercise: Marching to infinity



Show $f_n \to 0$ pointwise everywhere,

$f_n \overset{m}{\not\to} 0$

$f_n \not\to 0$ in $L^1(\mathbb{R})$

What about the "circular marching boxes" example in $[0,1]$?

<u>Exercise</u>: Uniqueness

If $f_n \xrightarrow{m} f$ & $f_n \xrightarrow{m} g$ then $f = g$ ae.

<u>Exercise</u>

Suppose $f_n \xrightarrow{n} f$ & $g_n \xrightarrow{m} g$.

a. Show that $f_n + g_n \xrightarrow{m} f + g$.

b. Show that if $\mu(X) < \infty$, then $f_n g_n \xrightarrow{m} fg$.

Hints: Proceed through cases.

Case 1: $f = g = 0$ ae.

Case 2: $g_n = g \ \forall n$

Case 3: arbitrary f_n, g_n

c. Show that part b can fail if μ is not finite.