# REAL ANALYSIS LECTURE NOTES:

# 2.4 MODES OF CONVERGENCE

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## 2.4.1 The relation between convergence in measure and pointwise convergence

Although convergence in measure does not imply pointwise convergence, we do have the following weaker (but still very useful) conclusion.

**Theorem 1.** If $f_n \xrightarrow{\text{m}} f$, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ pointwise a.e.

*Proof.* Since $f_n \xrightarrow{\text{m}} f$, we can find $n_1 < n_2 < \cdots$ such that

$$\forall\, n \ge n_k, \quad \mu\Big\{|f - f_n| > \frac{1}{k}\Big\} \;\le\; \frac{1}{2^k}.$$

Define

$$E_k \;=\; \Big\{|f - f_{n_k}| > \frac{1}{k}\Big\} \qquad \text{and} \qquad H_m \;=\; \bigcup_{k=m}^{\infty} E_k.$$

Then we have

$$\mu(E_k) \;<\; \frac{1}{2^k} \qquad \text{and} \qquad \mu(H_m) \;\le\; \sum_{k=m}^{\infty} \frac{1}{2^k} \;=\; \frac{1}{2^{m-1}}.$$

Set

$$Z \;=\; \bigcap_{m=1}^{\infty} H_m.$$

Then $\mu(Z) \le \mu(H_m) \le 1/2^{m-1}$ for every $m$, so we have $\mu(Z) = 0$.

If $x \notin Z$, then $x \notin H_m$ for some $m$. Hence $x \notin E_k$ for all $k \ge m$, which implies

$$|f(x) - f_{n_k}(x)| \;\le\; \frac{1}{k}, \quad \text{all } k \ge m.$$

Thus $f_{n_k}(x) \to f(x)$ for all $x \notin Z$. Since $Z$ has measure zero, we therefore have pointwise convergence of $f_{n_k}$ to $f$ almost everywhere. □

As an important special case we have the following.

**Corollary 2.** If $f_n \to f$ in $L^1(X)$, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ pointwise a.e.

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These notes follow and expand on the text "Real Analysis: Modern Techniques and their Applications," 2nd ed., by G. Folland.

### 2.4.2 A Cauchy criterion for convergence in measure

Although convergence in measure is not associated with a particular norm, there is still a useful Cauchy criterion for convergence in measure.

**Definition 3.** Given measurable $f_n$ on $X$, we say that $\{f_n\}_{n\in\mathbb{Z}}$ is *Cauchy in measure* if

$$\forall\, \varepsilon > 0, \quad \mu\{|f_m - f_n| \ge \varepsilon\} \to 0 \quad \text{as } m, n \to \infty.$$

Precisely, this means that

$$\forall\, \varepsilon, \eta > 0, \quad \exists\, N > 0 \text{ such that } m, n > N \implies \mu\{|f_m - f_n| \ge \varepsilon\} < \eta.$$

The usefulness of the Cauchy criterion is that it does not require us to know what the limit function is — we can test for Cauchyness without knowing whether the sequence converges. Further, the next theorem says that for convergence in measure, Cauchyness is equivalent to convergence. In order to prove the theorem, we need the following exercise.

**Exercise 4.** Let $f_n$ be measurable functions on $X$.

(a) Prove that

$$E = \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

is measurable.

(b) Show that

$$f(x) = \begin{cases} \lim\limits_{n\to\infty} f_n(x), & \text{if the limit exists,} \\ 0, & \text{otherwise,} \end{cases}$$

is a measurable function.

**Theorem 5.** Given measurable $f_n$ on $X$, the following statements are equivalent.

(a) $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure.

(b) There exists an $f$ such that $f_n \xrightarrow{\mathrm{m}} f$.

*Proof.* (b) $\Rightarrow$ (a). Exercise.

(a) $\Rightarrow$ (b). Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure.

Exercise: Show that there exists a subsequence $\{g_j\}_{j\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ such that

$$\forall\, j \in \mathbb{N}, \quad \mu\Big\{|g_j - g_{j+1}| \ge \frac{1}{2^j}\Big\} \le \frac{1}{2^j}.$$

Define

$$E_j = \Big\{|g_j - g_{j+1}| \ge \frac{1}{2^j}\Big\} \qquad \text{and} \qquad H_k = \bigcup_{j=k}^{\infty} E_j.$$

Then we have

$$\mu(E_j) \le \frac{1}{2^j} \qquad \text{and} \qquad \mu(H_k) \le \sum_{j=k}^{\infty} \frac{1}{2^j} = \frac{1}{2^{k-1}}.$$

If we set

$$Z = \bigcap_{k=1}^{\infty} H_k = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j = \limsup_{j\to\infty} E_j,$$

then we have $Z \subseteq H_k$ for every $k$, so $\mu(Z) = 0$.

Now fix any $x \notin Z$. Then $x \notin H_k$ for some $k$, and therefore $x \notin E_j$ for all $j \geq k$. Hence, for any $\ell \geq j \geq k$ we have

$$|g_j(x) - g_\ell(x)| \leq \sum_{i=j}^{\ell-1} |g_{i+1}(x) - g_i(x)| \leq \sum_{i=j}^{\ell-1} \frac{1}{2^i} \leq \frac{1}{2^{j-1}}.$$

As a consequence, the sequence of *scalars* $\{g_j(x)\}_{j\in\mathbb{N}}$ is Cauchy. Since $\mathbb{R}$ and $\mathbb{C}$ are complete, every Cauchy sequence of scalars converges. Hence, if we define

$$f(x) = \begin{cases} \lim\limits_{j\to\infty} g_j(x), & \text{if the limit exists,} \\ 0, & \text{otherwise,} \end{cases}$$

then $f$ is measurable by Exercise 4, and since the limit exists for every $x \notin Z$ we have that $g_j \to f$ pointwise a.e. (if $\mu$ is complete, measurability of $f$ is easier, and in fact we could define it any way we like on $Z$ in that case).

Now will show that $g_j$ converges in measure to $f$. Fix any $j$. If $x \notin H_j$, then, as above, we have that $|g_j(x) - g_\ell(x)| \leq \frac{1}{2^{j-1}}$ for all $\ell \geq j$. Hence

$$|g_j(x) - f(x)| = \Big|g_j(x) - \lim_{\ell\to\infty} g_\ell(x)\Big| = \lim_{\ell\to\infty} |g_j(x) - g_\ell(x)| \leq \frac{1}{2^{j-1}}.$$

Therefore

$$\Big\{|g_j - f| \geq \frac{1}{2^{j-1}}\Big\} \subseteq H_k,$$

and so

$$\mu\Big\{|g_j - f| \geq \frac{1}{2^{j-1}}\Big\} \leq \mu(H_j) \leq \frac{1}{2^{j-1}} \to 0 \quad \text{as } j \to \infty.$$

It follows from this that $g_j \xrightarrow{\text{m}} f$.

Thus, we have shown that $\{g_j\}_{j\in\mathbb{N}}$ is a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ that converges in measure to $f$. Exercise: Combine this with the fact that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure to show that $f_n \xrightarrow{\text{m}} f$. □

As an application, we can use the Cauchy criterion for convergence in measure to show that $L^1(X)$ is a Banach space, i.e., a complete normed space (see the review of metrics, norm, and convergence for the definition of a complete space).

**Theorem 6** (Completeness of $L^1(X)$)**.** $L^1(X)$ is a Banach space.

*Proof.* We already know that $L^1(X)$ is a normed space (once we identify functions that are equal a.e.), so we just have to show that it is complete. This means that we have to show that every Cauchy sequence in $L^1(X)$ converges.

So, suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(X)$. This means that

$$\forall\, \varepsilon > 0, \quad \exists N > 0 \text{ such that} \quad m, n > N \implies \|f_m - f_n\|_1 < \varepsilon.$$

Exercise: Show that this implies that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure (very similar to the exercise that shows that convergence in $L^1(X)$ implies convergence in measure — use Tchebyshev's inequality).

By Theorem 5, this implies that there exists a measurable function $f$ such that $f_n \xrightarrow{\text{m}} f$. Hence there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges to $f$ pointwise a.e.

Fix now any $\varepsilon > 0$. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^1(X)$, there exists an $N > 0$ such that

$$m, n > N \implies \|f_m - f_n\|_1 < \varepsilon.$$

Fix any $k$ such that $n_k > N$. Then by Fatou's Lemma, we have that

$$\begin{aligned}
\|f - f_{n_k}\|_1 &= \int |f - f_{n_k}| \\
&= \int \lim_{j\to\infty} |f_{n_j} - f_{n_k}| \\
&\le \liminf_{j\to\infty} \int |f_{n_j} - f_{n_k}| \\
&= \liminf_{j\to\infty} \|f_{n_j} - f_{n_k}\|_1 \le \varepsilon.
\end{aligned}$$

Thus $f_{n_k} \to f$ in $L^1(X)$.

Exercise: Combine this with the fact that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^1(X)$ to show that $f_n \to f$ in $L^1(X)$. □

### 2.4.4 Egoroff's Theorem

In general, pointwise convergence does not imply convergence in measure. However, for a finite measure space, this is true, and in fact we will see in this section that much more is true. Quoting from Royden's text "Real Analysis," he quotes the mathematician Littlewood's "Three Principles." Weaving quote (roman text) with comments (italics), this is:

> The extent of knowledge required is nothing like so great as is sometime supposed. There are three principles, roughly expressible in the following terms:
>
> *(a)* Every (measurable) set is nearly a finite union of intervals *(because it is nearly an open set)*;
>
> *(b)* Every (measurable) function is nearly continuous *(this is Lusin's Theorem)*;
>
> *(c)* Every convergent sequence of (measurable) functions is nearly uniformly convergent *(this is Egoroff's Theorem)*.

See Royden for the full quote, which is quite interesting. In this section we deal with Littlewood's third principle, Egoroff's Theorem.

**Theorem 7** (Egoroff's Theorem)**.** Suppose that $\mu$ is a finite measure, and that $f_n, f\colon X \to \mathbb{C}$ are measurable. If $f_n \to f$ pointwise a.e., then for every $\varepsilon > 0$ there exists a measurable $E \subseteq X$ such that

(a) $\mu(E) < \varepsilon$, and

(b) $f_n$ converges uniformly to $f$ on $E^{\mathrm{C}}$, i.e.,

$$\lim_{n\to\infty} \sup_{x\notin E} |f(x) - f_n(x)| \;=\; 0.$$

*Proof.* Let $Z$ be the set of measure zero where $f_n(x)$ does not converge to $f(x)$. For $k, n \in \mathbb{N}$, define the measurable sets

$$E_n(k) \;=\; \bigcup_{m=n}^{\infty} \Bigl\{|f - f_m| \geq \frac{1}{k}\Bigr\} \qquad \text{and} \qquad Z_k \;=\; \bigcap_{n=1}^{\infty} E_n(k).$$

Now, if $x \in Z_k$, then $x \in E_n(k)$ for every $n$. Hence, for each $n$ there must exist an $m \geq n$ such that $|f(x) - f_m(x)| > \frac{1}{k}$. Therefore $f_n(x)$ does not converge to $f(x)$, so $x \in Z$. Thus

$$Z_k \;\subseteq\; Z,$$

and therefore $\mu(Z_k) = 0$ by monotonicity. Since $E_1(k) \supseteq E_2(k) \supseteq \cdots$, we therefore have by continuity from above that

$$\lim_{n\to\infty} \mu(E_n(k)) \;=\; \mu(Z_k) \;=\; 0.$$

Choose now any $\varepsilon > 0$. Then for each $k$, we can find an $n_k$ such that

$$\mu(E_{n_k}(k)) \;<\; \frac{\varepsilon}{2^k}.$$

Define

$$E \;=\; \bigcup_{k=1}^{\infty} E_{n_k}(k),$$

then we have by subadditivity that $\mu(E) \leq \varepsilon$. And if $x \notin E$, then $x \notin E_{n_k}(k)$ for every $k$, and therefore $|f(x) - f_m(x)| < \frac{1}{k}$ for all $m \geq n_k$. Thus, we have shown that for each $k \in \mathbb{N}$, there exists an $n_k > 0$ such that for all $m \geq n_k$ we have

$$\sup_{x\notin E} |f(x) - f_m(x)| \;\leq\; \frac{1}{k}.$$

This implies that $f_n$ converges uniformly to $f$ on $E^{\mathrm{C}}$. □

### 2.4.5 Lusin's Theorem

Lusin's Theorem is a result about Lebesgue measure.

**Theorem 8** (Lusin's Theorem)**.** Given a measurable set $E \subseteq \mathbb{R}^d$ and given $f\colon E \to \mathbb{C}$, the following statements are equivalent.

(a) $f$ is measurable.

(b) For each $\varepsilon > 0$, there exists a closed set $F \subseteq E$ with $|E \setminus F| < \varepsilon$ such that $f|_F$ is continuous, i.e.,

$$\forall\, x_k, x \in F, \quad x_k \to x \implies f(x_k) \to f(x).$$

*Proof.* (a) $\Rightarrow$ (b). First we prove the result for simple functions. Suppose that $\phi = \sum_{j=1}^N a_j \chi_{E_j}$ is a simple function, and that the $E_j$ are disjoint. Fix $\varepsilon > 0$. Since $E_j$ is measurable, there exists a closed $F_j \subseteq E_j$ such that

$$|E_j \setminus F_j| \; < \; \frac{\varepsilon}{n}, \quad j = 1, \dots, n.$$

Then

$$F \; = \; \bigcup_{j=1}^{n} E_j$$

is closed, and $|E \setminus F| < \varepsilon$.

If $E$ is a bounded set, then the $F_j$ are compact, and hence

$$\operatorname{dist}(F_j, F_k) \; > \; 0$$

if $j \neq k$. Since $\phi$ is constant on each $F_j$, it follows that $\phi|_F$ is continuous.

Exercise: Extend to the case where $E$ is not bounded by considering the sets

$$E_k \; = \; \{x \in E : \|x\| \leq k\}.$$

Now let $f\colon E \to \mathbb{C}$ be an arbitrary measurable function. Let $\phi_n$ be simple functions such that $\phi_n(x) \to f(x)$ for each $x \in E$. Fix $\varepsilon > 0$. By the previous case, for each $n$ we can find a closed $F_n \subseteq E$ such that

$$|E \setminus F_n| \; < \; \frac{\varepsilon}{2^{n+1}}$$

and $\phi_n|_{F_n}$ is continuous.

Suppose that $E$ is bounded. Then by Egoroff's Theorem, there exists a closed $F_0 \subseteq E$ such that

$$|E \setminus F_0| < \frac{\varepsilon}{2}$$

and $f_n$ converges to $f$ uniformly on $F_0$. Define

$$F \; = \; \bigcap_{n=0}^{\infty} F_n.$$

Then $F$ is closed since each $F_n$ is closed, and

$$|E \setminus F| \; = \; \left| \bigcup_{n=0}^{\infty} (E \setminus F_n) \right| \; \leq \; \sum_{n=0}^{\infty} |E \setminus F_n| \; \leq \; \varepsilon.$$

Since $\phi_n|_{F_n}$ is continuous, $\phi_n|_F$ is continuous as well. And since $\phi_n$ converges to $f$ uniformly on $F$, we have that $f|_F$ is continuous. This completes the proof for the case that $E$ is bounded.

Exercise: Extend to the case where $E$ is unbounded by considering the sets

$$E_k \; = \; \{x \in E : k - 1 \leq \|x\| < k\}.$$

(b) $\Rightarrow$ (a). Suppose that statement (b) holds. By considering the real and imaginary parts of $f$ separately, it suffices to assume that $f$ is real-valued.

By hypothesis, for each $n \in \mathbb{N}$ there exists a closed $F_n \subseteq E$ such that

$$|E \setminus F_n| \;<\; \frac{1}{n}$$

and $f|_{F_n}$ is continuous. Set

$$H \;=\; \bigcup_{n=1}^{\infty} F_n.$$

Then $H$ is an $F_\sigma$-set, so is measurable. Also, for every $n$ we have that

$$|E \setminus H| \;\le\; |E \setminus F_n| \;<\; \frac{1}{n}|,$$

so $|E \setminus H| = 0$. Therefore we can write $E = H \cup Z$ where $Z$ has measure zero and is disjoint from $H$.

If we fix any $a \in \mathbb{R}$, then we have that

$$\begin{aligned} \{f > a\} \;&=\; \{x \in H : f(x) > a\} \,\cup\, \{x \in Z : f(x) > a\} \\ &=\; \bigcup_{n=1}^{\infty} \{x \in F_n : f(x) > a\} \,\cup\, \{x \in Z : f(x) > a\}. \end{aligned}$$

Since each $f|_{F_n}$ is continuous, we have that $\{x \in F_n : f(x) > a\}$ is relatively open with respect to $F_n$ (i.e., it is the intersection of an open set $U \subseteq \mathbb{R}^d$ with $F_n$) and hence is measurable. And since Lebesgue measure is complete, we know that $\{x \in Z : f(x) > a\}$ is measurable. Therefore we conclude that $\{f > a\}$ is measurable. Since this is true for every real number $a$, we have shown that $f$ is a measurable function. $\square$