2.5 Product Measures

We want to develop measures & integration on $X \times Y$.

Suppose that $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ are measure spaces.

We want to construct an appropriate measure space $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$.

What sets will be in $\mathcal{M} \otimes \mathcal{N}$ & what will their measures be? We'll sketch the construction.

Start with what we certainly want to be in $\mathcal{M} \otimes \mathcal{N}$.

If $A \in \mathcal{M}$, $B \in \mathcal{N}$, then we certainly want $A \times B$ to be in $\mathcal{M} \otimes \mathcal{N}$. So $\mathcal{M} \otimes \mathcal{N}$ should include all such sets.

**Notation:** we call $A \times B$ a rectangle, even though $A \times B$ is clearly an abuse of terminology.

\[ \begin{array}{ccc}
\mathcal{I} & | & A \times B \\
\hline
A & & \\
\end{array} \]
We declare that

\[(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B).\]

So now we know how to measure rectangles, and we have to figure out how to extend to arbitrary \(\mathcal{M}\) sets.

If we set

\[\mathcal{E} = \{ A \times B : A \in \mathcal{M}, B \in \mathcal{N} \},\]

then we define

\[\mathcal{M} \odot \mathcal{N} = \mathcal{M}(\mathcal{E}), \text{ a } \sigma\text{-algebra generated by } \mathcal{E}.\]

Exercise:

We don't even need all \(\mathcal{E}\) rectangles. Show that \(\mathcal{M} \odot \mathcal{N}\) is generated by

\[\{ A \times Y : A \in \mathcal{M} \} \cup \{ X \times B : B \in \mathcal{N} \}\]

\[X \times B = \pi_2^{-1}(B)\]
Now we construct a measure $\mu \times \nu$ by going through a pre-measure construction.

First we form an algebra by taking all finite unions of disjoint algebras:

$$A = \{ \bigcup_{j=1}^{n} A_j \times B_j : A_j \in \mathcal{M}, B_j \in \mathcal{N}, \ A_j \times B_j \text{ disjoint} \}$$

To show this is an algebra, just show that $\mathcal{E}$ is an elementary family, i.e.

a. $\emptyset \in \mathcal{E}$

b. $E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$

c. $E \in \mathcal{E} \Rightarrow E^c = \bigcup_{j=1}^{n} E_j, \ E_j \in \mathcal{M} \text{ disjoint}$
\[(A \times B) \cap = (A^c \times Y) \cup (A \times B^c) \text{ disjointly.}\]
If \( E = \bigcup_{j=1}^{\hat{n}} A_j \times B_j \in A \) (disjoint sum of rectangles), then we set

\[
\rho(E) = \sum_{j=1}^{\hat{n}} \mu(A_j) \cdot \nu(B_j)
\]

This defines a premeasure. We then have an outer measure

\[
\rho^*(E) = \inf \left\{ \sum_k \rho(E_k) : E_k \in A, E = \bigcup E_k \right\}
\]

Since we've gone through the premeasure process, every set in \( M \otimes N \) is \( \rho^* \)-measurable, and

\[
\rho^*(A \times B) = \mu(A) \cdot \nu(B).
\]

We then define

\[
\mu \times \nu = \rho^*|_{M \otimes N}
\]

This is a measure with \( \sigma \)-algebra \( M \otimes N \).

Good news:

\[
(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)
\]

\( \forall A \in M, B \in N \).
In fact, \( \mu \times v \) is the unique measure on \( M \times N \) for which \( (\mu \times v)(A \times B) = \mu(A) v(B) \) holds.

By induction, we can extend to higher-fold products, and show that associativity holds, e.g.,
\[
M_1 \otimes (M_2 \otimes M_3) = (M_1 \otimes M_2) \otimes M_3
\]
\[
\mu_1 \times (\mu_2 \times \mu_3) = (\mu_1 \times \mu_2) \times \mu_3
\]

We’ll take the construction of \( \mu \times v \) as given from now on.
Now we proceed towards theorems involving repeated integration.

**Sections of a Set**

Given $E \subseteq X \times Y$, $x \in X$, and $y \in Y$,

- A **$x$-section** of $E$ is $E_x = \{y \in Y : (x, y) \in E\}$.
- A **$y$-section** of $E$ is $E^y = \{x \in X : (x, y) \in E\}$.

Note that $E_x \subseteq Y$ and $E^y \subseteq X$. 
Exercises

a. \((E^c)^c_x = (E_x)^c\), \((E^c)^y = (E^y)^c\)

b. \((UE_j)^c_x = U(E_j)^c_x\), \((UE_j)^y = U(E_j)^y\)

c. \((A \times B)^c_x = \{B, \ x \in A\}
\{\emptyset, \ x \notin A\}\)

\((A \times B)^y = \{A, \ y \in B\}
\{\emptyset, \ y \notin B\}\)

Now we show that sections of a \(\mathfrak{M}\) set are \(\mathfrak{M}\).

Theorem
If \(E \in \mathfrak{M} \otimes \mathfrak{N}\) then \(E_x \in \mathfrak{N}\) & \(E^y \in \mathfrak{M}\) \(\forall x \in X, y \in Y\).

Proof:
Let
\(\mathcal{R} = \{E \in X \times Y : E_x \in \mathfrak{N} \land E^y \in \mathfrak{M} \ \forall x \in X, y \in Y\}\)

Exercise: Use the exercise above to show that \(\mathcal{R}\) is a \(\mathfrak{M}\)-algebra, and that \(A \times B \in \mathcal{R}\) whenever \(A \in \mathfrak{M}, B \in \mathfrak{N}\). Hence \(\mathcal{R}\) contains
all rectangles, and therefore contains a $\sigma$-algebra generated by rectangles, which is $M \otimes N$. That is, $M \otimes N \leq \mathcal{Q}$, which proves the theorem.

**Sections of a function**

Given $f : X \times Y \to Z$, $x \in X$, $y \in Y$:

- The **$x$-section of $f$** is
  \[
  f_x : Y \to Z \quad \text{i.e., } \quad f_x(y) = f(x, y)
  \]

- The **$y$-section of $f$** is
  \[
  f_y : X \to Z \quad \text{i.e., } \quad f_y(x) = f(x, y)
  \]

**Note**

\[
(X_E)_x = X_{E_x} \quad \& \quad (X_E)^y = X_{E^y}.
\]
Exercise
Show that if $f$ is $M \otimes N - \mathfrak{m}$, then

$f \chi_x$ is $N - \mathfrak{m}$ \quad \forall x \in X \quad \text{and} \\
$f \chi_y$ is $M - \mathfrak{m}$ \quad \forall y \in Y$.

Motivation: Characteristic Functions

Suppose for the moment that we could switch integrals at will. Then we would have

$$(\mu \times \nu)(E) = \int \int \chi_{E}(x,y) \; d(\mu \times \nu)(x,y)$$

$$= \int \int \chi_{E}(y) \; d\mu(x) \; d\nu(y)$$

$$= \int \left( \int \chi_{E}(y) \; d\nu(y) \right) \; d\mu(x)$$

$$= \int \nu(E_x) \; d\mu(x).$$

The next theorem will show directly that

$$(\mu \times \nu)(E) = \int \nu(E_x) \; d\mu(x) \quad \text{is valid}.$$
We begin with finite measures.

**Theorem**
Suppose \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) are finite measures.

If \(E \in \mathcal{M} \otimes \mathcal{N}\), then the following statements hold:

a. \(x \mapsto \nu(E_x)\) is \(\mathcal{M}\)-measurable

b. \(y \mapsto \mu(E^y)\) is \(\mathcal{N}\)-measurable

c. \((\mu \times \nu)(E) = \int \nu(E_x) \, d\mu(x) = \int \mu(E^y) \, d\nu(y)\)

**Proof:**
Let \(C\) be the set of all \(E \in \mathcal{M} \otimes \mathcal{N}\) for which a, b, and c hold.

Claim 1: \(C\) contains all measurable rectangles.

To see this, suppose \(E = A \times B\) where \(A \in \mathcal{M}\), \(B \in \mathcal{N}\).
Then \( E_x \) is either \( B \) or \( \emptyset \), which are \( \emptyset \), \( \emptyset \).
Likewise \( E^y \) is \( \emptyset \). Also,
\[
\nu(E_x) = \chi_A(x) \cdot \nu(B)
\]
so
\[
\int \nu(E_x) \, d\mu(x) = \int \chi_A(x) \, \nu(B) \, d\mu(x)
\]
\[
= \mu(A) \cdot \nu(B)
\]
\[
= (\mu \times \nu)(E)
\]
and similarly \( \int \mu(E^y) \, d\nu(y) = (\mu \times \nu)(E) \).
Hence \( E \in \mathcal{C} \).

Exercise: Extend \( A \) to show that \( A \subseteq \mathcal{C} \).
Claim 2: \( E \) is closed under increasing unions

Suppose \( E_1 \subseteq E_2 \subseteq \ldots \) belong to \( E \), and set \( E = \bigcup E_k \). Define

\[
f_k(y) = \mu(E_k^y), \quad y \in Y.
\]

Since \( E_k \in E \), we have that \( f_k \) is \( \ominus \). Since \( E_1^y \subseteq E_2^y \subseteq \ldots \) and \( \bigcup E_k^y = E^y \), we have by continuity from below that \( f_k \) converge:

\[
f(y) = \mu(E^y) = \lim_{k \to \infty} \mu(E_k^y) = \lim_{k \to \infty} f_k(y).
\]

Hence \( f \) is \( \ominus \). Also,

\[
\int \mu(E^y) \, d\nu(y) = \lim_{k \to \infty} \int \mu(E_k^y) \, d\nu(y) \quad \text{MCT}
\]

\[
= \lim_{k \to \infty} (\mu \times \nu)(E_k) \quad \text{since } E_k \in E
\]

\[
= (\mu \times \nu)(E) \quad \text{by continuity from below.}
\]

The remaining properties are symmetric, so \( E \in E \).
Claim 3: $E$ is closed under decreasing intersections.

Suppose $E_1 \supseteq E_2 \supseteq \ldots$ belong to $E$, and set $E = \bigcap E_k$. As before, $f_k(x) = \mu(E_k^y)$ is $\mathbb{R}$.

Also $E_1^y \supseteq E_2^y \supseteq \ldots$ & $E^y = \bigcap E_k^y$. Since our measures are finite, we have by continuity from above that

$$f(y) = \mu(E^y) = \lim_{k \to \infty} \mu(E_k^y) = \lim_{k \to \infty} f_k(y)$$

converges, & therefore is $\mathbb{R}$. Also, $f_k(y) \geq f(y) \geq 0$ and all these functions are integrable, since

$$\int f_i(y) \, d\nu(y) = \int \mu(E_i^y) \, d\nu(y)$$

$$\leq \int \mu(x) \, d\nu(y)$$

$$= \mu(x) \cdot \nu(Y) < \infty.$$

Exercise: $f_k \uparrow f \geq 0$ & $f_i \in L^1(Y) \Rightarrow \int f_k \to \int f.$

Therefore we have.
\[ \int_{E} (x^3) \, d\lambda(y) = \lim_{k \to \infty} \int_{E_k} (x^3) \, d\nu(y) \] by the exercise

= \lim_{k \to \infty} (\mu \times \nu)(E_k) \] since \( E_k \in \mathcal{E} \)

= (\mu \times \nu)(E) \] by continuity from above

Thus \( E \in \mathcal{E} \).

Now we appeal to the Monotone Class Lemma: because

i. \( \mathcal{A} \) is an algebra & \( \mathcal{A} \subseteq \mathcal{E} \),

ii. \( \mathcal{E} \) is closed under increasing unions,

iii. \( \mathcal{C} \) is closed under decreasing intersections,

it follows that \( \mathcal{C} \) contains \( \mathcal{A} \), the \( \sigma \)-algebra generated by \( \mathcal{A} \), which is \( M \otimes N \).

Remark

For a proof of the Monotone Class Lemma, see Folland's text. It actually says that if \( \mathcal{I} \) is the smallest monotone class (i.e., satisfying i., ii., iii.) that contains \( \mathcal{A} \),

then \( \mathcal{I} \) is the \( \sigma \)-algebra generated by \( \mathcal{A} \). Hence we conclude that \( \mathcal{C} \subseteq \mathcal{I} = M \otimes N \).
Exercise: Show that the preceding theorem remains valid if we only assume that $\Omega, \nu$ are $\sigma$-finite.

Example (Zygmund)

Let $X = Y = [0, 1]$. Place countably many squares on the diagonal of the unit square:

![Diagram showing squares on the diagonal of a unit square]

Set $c_n = \frac{1}{|Q_n|}$. Define $f(x,y)$ to be zero outside $UQ_n$. On each $Q_n$ let $f$ have values as follows:

<table>
<thead>
<tr>
<th></th>
<th>$-c_n$</th>
<th>$c_n$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$c_n^*$</td>
<td>$-c_n^*$</td>
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</tbody>
</table>

$Q_n$
Then \( f \) is \( 0 \) &
\[
\int_0^1 \int_0^1 f(x, y) \, dy \, dx = 0 = \int_0^1 \int_0^1 f(x, y) \, dx \, dy
\]

Hence the following iterated integrals exist:
\[
\int_0^1 \left( \int_0^1 f(x, y) \, dy \right) \, dx = 0 = \int_0^1 \left( \int_0^1 f(x, y) \, dx \right) \, dy.
\]

However,
\[
\int_{[0, 1]^2} f^+(x, y) \, (dx \times dy) = \sum_{k=1}^{a^2} \frac{1}{2} \frac{1}{|Q_n|} |Q_n| = 0
\]

and likewise \( \int_{[0, 1]^2} f^- = \infty \). Hence the double integral
\[
\int_{[0, 1]^2} f(x, y) \, (dx \times dy)
\]
does not exist.
Exercise
Show that
\[ \int_1^\infty \left( \int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2} \, dx \right) \, dy = -\frac{\pi}{4} \]

while
\[ \int_1^\infty \left( \int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2} \, dy \right) \, dx = \frac{\pi}{4} \]

Hint: Use the fact that \( \frac{\partial}{\partial x^2+y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \).

Thus, in general, iterated integrals
\[ \int \left( \int f(x,y) \, dx \right) \, dy \]
\[ \int \left( \int f(x,y) \, dy \right) \, dx \]

and the double integral
\[ \iint f(x,y) \, d(x+y)(x,y) \]

need not be equal. Tonelli's and Fubini's Theorems will provide hypotheses under which these are all equal.
Tonelli's Theorem says that interchange is allowed for nonnegative functions (if the measure spaces are σ-finite).

Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be σ-finite measure spaces.

If \(f: X \times Y \to [0, \infty] \) is \(\mathcal{M} \times \mathcal{N}\)-measurable, then the following statements hold:

a. \(f_x(y) = f(x, y) \) is \(\mathcal{M}\) for \(x \in X\).

b. \(f^y(x) = f(xy) \) is \(\mathcal{N}\) for \(y \in Y\).

c. \(g(x) = \int f_x(y) \, d\nu(y) \) is \(\mathcal{M}\).

d. \(h(y) = \int f^y(x) \, d\mu(x) \) is \(\mathcal{N}\).

e. As extended real numbers,

\[
\int f(xy) \, d(\mu \times \nu)(x, y) = \int \left( \int f(xy) \, d\mu(x) \right) \, d\nu(y) = \int \left( \int f(x, y) \, d\nu(y) \right) \, d\mu(x).
\]

Proof:

Exercise: Verify that our previous results.
establish Tonelli's Theorem for the case where $f$ is a characteristic function, and extend $\mathbb{R}$ by linearity to simple functions.

Given an arbitrary \( f: X \times Y \to [0, \infty] \), let $\phi_n$ be simple functions $0 \leq \phi_n \uparrow f$.

Then $g_n(x) = \int \phi_n(x,y) \, d\nu(y)$ is \( \mathbb{M} \), and by the Monotone Convergence Theorem, for each $x$,

$$ g_n(x) = \int \phi_n(x,y) \, d\nu(y) \uparrow \int f(x,y) \, d\nu(y) = g(x), $$

so $g$ is \( \mathbb{M} \), and similarly $h$ is \( \mathbb{M} \). Also,

$$ \int f \, d\mu(x) = \lim_{n \to \infty} \int \phi_n \, d\mu(x) \quad (\text{MCT}) $$

$$ = \lim_{n \to \infty} \int \left( \int \phi_n(x,y) \, d\nu(y) \right) \, d\mu(x) $$

$$ = \lim_{n \to \infty} \int g_n \, d\mu $$

$$ = \int g \, d\mu \quad (\text{MCT}) $$

$$ = \int \int f(x,y) \, d\nu(y) \, d\mu(x) $$
and equality of the other iterated integral is similar.

The following corollary is extremely useful: to test whether \( f \in L^1(X \times Y) \), we can check any one of three integrals for finiteness.

**Corollary**

Let \( (X, M, \mu) \) & \( (Y, N, \nu) \) be \( \sigma \)-finite measure spaces.

If \( f: X \times Y \to \mathbb{R} \) or \( \mathbb{C} \) is \( \overline{\mathbb{C}} \), then (as extended real numbers)

\[
\int |f| \, d(\mu \times \nu) = \int |f(x,y)| \, d\mu(x) \, d\nu(y)
\]

\[
= \int |f(x,y)| \, d\nu(y) \, d\mu(x)
\]

Consequently, if any one of these integrals is finite, then \( f \in L^1(X \times Y) \).
Fubini's Theorem allows an interchange of integrals if $f$ is integrable (thereby again avoiding the ambiguity that is $\infty - \infty$).

**Fubini's Theorem**

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be $\sigma$-finite measure spaces.

If $f \in L^1(X \times Y)$, then the following statements hold:

a. $f_x \in L^1(Y)$ for $\mu$-a.e. $x \in X$,

b. $f_y \in L^1(X)$ for $\nu$-a.e. $y \in Y$.

c. $g(x) = \int f_x(y) \, d\nu(y)$ belongs to $L^1(X)$

d. $h(y) = \int f_y(x) \, d\mu(x)$ belongs to $L^1(Y)$

e. $\int f(x,y) \, d(\mu \times \nu)(x,y) = \int \left( \int f(x,y) \, d\mu(x) \right) \, d\nu(y)$

\[ = \int \left( \int f(x,y) \, d\nu(y) \right) \, d\mu(x) \]
Proof:

Suppose first that $f > 0$. Then by Tonelli's Theorem, $f_x, f_y, g, \text{ and } h$ are $\mathcal{M}$, and

$$0 \leq \int g \, d\mu = \int h \, d\nu = \iint f \, d(\mu \times \nu) < \infty.$$ 

Hence, $g \in L^1(X)$ and $h \in L^1(Y)$, since they are nonnegative. Consequently, $g$ and $h$ must be finite a.e., so

$$0 \leq g(x) = \int f_x \, d\nu < \infty \text{ a.e.} \ x$$

$$0 \leq h(y) = \int f_y \, d\mu < \infty \text{ a.e.} \ y$$

Therefore, $f_x$ is integrable for a.e. $x$, and $f_y$ is integrable for a.e. $y$. This completes the proof for the case $f > 0$.

For general $f \in L^1(X)$, write
\[ f = (f_1 - f_2) + i (f_3 - f_4) \]

where \( f_1, f_2, f_3, f_4 \). Then (exercise) apply the preceding case to each \( f_i \) & show that the result follows.

**Remark**
Technically, elements of \( L' \) are equivalence classes of functions that are equal a.e. Fubini's Theorem applies to any of the representatives of \( f \).

**Remark**
Again, remember the Corollary to Tonelli's Theorem, which tells us how to check whether \( f \in L'(X \times Y) \) we can check any one of three integrals, whichever is more convenient.
Unfortunately, \( \mu \times \nu \) is rarely complete, so sometimes we need a version of Tonelli/Fubini for the completion of \( \mu \times \nu \). For the general case, see Folland's text.

For example, Lebesgue measure on \( \mathbb{R}^n \) is the completion of the \( n \)-fold product of Lebesgue measures on \( \mathbb{R} \). The statement of Tonelli & Fubini for Lebesgue measure is as follows.

**Tonelli's Theorem**

Let \( E \subseteq \mathbb{R}^m \) & \( F \subseteq \mathbb{R}^n \) be Lebesgue measurable.

If \( f : E \times F \to [0, \infty] \) is \( \mathcal{m} \times \mathcal{m} \)-measurable,

\[
\begin{align*}
\text{a. } f^x & \in \mathcal{m} \quad \forall x \in E \\
\text{b. } f^y & \in \mathcal{m} \quad \forall y \in F \\
\text{c. } g(x) &= \int_F f_x(y) \, dy \quad \text{is } \mathcal{m} \text{ on } E \\
\text{d. } h(y) &= \int_E f^y(x) \, dx \quad \text{is } \mathcal{m} \text{ on } F \\
\text{e. } \int_{E \times F} f(x, y) \, dx \, dy &= \int_E \left( \int_F f(x, y) \, dx \right) \, dy \\
&= \int_F \left( \int_E f^y(x) \, dy \right) \, dx
\end{align*}
\]
Fubini's Theorem

Let $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ be Lebesgue measurable.

If $f \in L^1(ExF)$, then:

a. $f_x \in L^1(E)$ for a.e. $x \in E$

b. $f_y \in L^1(E)$ for a.e. $y \in F$

c. $g(x) = \int_F f(x,y) \, dy$ belongs to $L^1(E)$

d. $h(y) = \int_E f(x,y) \, dx$ belongs to $L^1(F)$

e. $\int_{ExF} f(x,y) \, d(x,y) = \int_E (\int_F f(x,y) \, dy) \, dx$

= $\int_E (\int_F f(x,y) \, dy) \, dx$
The Integral as Area under the Graph

Exercise (see Folland 25 #50)

Let \( (X, \mathcal{M}, \mu) \) be a \( \sigma \)-finite measure space, & let \( f: X \to [0, \infty] \) be \( \mu \)-measurable.

a. The region under the graph of \( f \) is

\[ G_f = \{(x, y) \in X \times [0, \infty] : 0 \leq y \leq f(x)\} \]

Show that \( G_f \) is \( M \times \mathcal{B} \)-measurable \( (\mathcal{B} = \text{Borel } \sigma\text{-algebra in } \mathbb{R}) \)

and that

\[ (\mu \times m)(G_f) = \int f \, d\mu \]

\( (m = \text{Lebesgue measure}) \).

See Folland for a hint.

b. The graph of \( f \) is

\[ \Gamma_f = \{(x, f(x)) : x \in X\} \subseteq X \times [0, \infty] \]

Show that \( \Gamma_f \) is \( \mathcal{M} \), & \( (\mu \times m)(\Gamma_f) = 0 \).
Hint: Consider

\[ E_n = \{ \exists n \leq f < \varepsilon(n+1) \} \quad \& \quad E_\infty = \{ f = \infty \} \]

Convolution

Let \( f, g : \mathbb{R} \to \mathbb{R} \) be Lebesgue \( \mathcal{C} \).

a. Show that \( f(x)g(y) \) is Lebesgue \( \mathcal{C} \) on \( \mathbb{R}^2 \).

b. Show \( f(x-y)g(y) \) is \( \mathcal{C} \) on \( \mathbb{R}^2 \).

Hint: Composition of part a with a linear transformation.

c. Given \( f, g \in L^1(\mathbb{R}) \), show their convolution

\[
(f \ast g)(x) = \int f(x-y)g(y) \, dy
\]

is defined a.e. \( L \) is \( \mathcal{C} \).

Hint: Show \( \int \int |f(x-y)g(y)| \, dx \, dy < \infty \), apply Fubini.

d. Show \( f \ast g \in L^1(\mathbb{R}) \) \& \( \|f \ast g\|_1 \leq \|f\|_1 \|g\|_1 \).

(Thus \( L^1(\mathbb{R}) \) is a Banach algebra under convolution.)