2.6 The n-dimensional Lebesgue integral

We've already developed Lebesgue measure directly on IR^n, so we won't do it again. Folland's approach is to define 1-dimensional Lebesgue measure as a special case of Lebesgue-Stieltjes measures, and then create n-dimensional Lebesgue measure as an n-fold product of 1-dimensional Lebesgue measures. The results are the same.

There are a few important issues about Lipschitz transformations that we have not covered yet, so we will do those now.

**Definition**

A function \( T: \mathbb{R}^n \to \mathbb{R}^n \) is **Lipschitz** if

\[
\exists C > 0 \text { s.t. } \forall x, y \in \mathbb{R}^n, \quad || T(x) - T(y) || \leq C || x - y ||.
\]

**Remark**

All norms on \( \mathbb{R}^n \) are equivalent since \( \mathbb{R}^n \) is finite-dimensional. Hence if \( T \) is Lipschitz w.r.t. one norm, say the Euclidean norm, then it Lipschitz w.r.t. another norm, say \( C' \)-norm. The constant \( C \) will change, but it will still exist. So, we can use whatever norm is most convenient.
Exercise

a. Show that Lipschitz functions are uniformly continuous.

b. Show that if \( f: \mathbb{R} \to \mathbb{R} \) is differentiable and \( f' \) is bounded, then \( f \) is Lipschitz.
   
   Hint: MVT

c. Give an example of a function \( f \) that is Lipschitz but not differentiable everywhere.

d. Show that \( f(x) = x^2 \) is not Lipschitz, but it is locally Lipschitz, i.e., Lipschitz on every compact set.

e. Show that every linear function \( T: \mathbb{R}^n \to \mathbb{R}^n \) is Lipschitz.

f. The translation map \( T_\alpha: \mathbb{R}^n \to \mathbb{R}^n \)
   
   \[ x \mapsto x + \alpha \]

   is not linear, but is Lipschitz.
Exercise
Show that Lipschitz mappings preserve Lebesgue measurability.

If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is Lipschitz & \( E \subseteq \mathbb{R}^n \) is \( C \),
then \( T(E) \) is \( C \).

Hints: \( T \) is continuous, and therefore maps compact sets to compact sets. Every \( C \) \( E \) can be written \( E = F \cup Z \) where \( F \) is an \( F_0 \)-set and \( |Z| = 0 \). Show \( |T(E)| = 0 \).

Theorem
If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is linear and \( E \subseteq \mathbb{R}^n \) is \( C \),
then
\[
|T(E)| = |\det(T)| \cdot |E|.
\]

Proof:
Let \( d = |\det(T)| \).

Given a \( C \) \( E \subseteq \mathbb{R}^n \) & \( \varepsilon > 0 \), find cubes \( Q_k \)
with \( E \subseteq \cup Q_k \) and
\[
|E| \leq \sum_k |Q_k| \leq |E| + \varepsilon.
\]

Each \( T(Q_k) \) is a parallelepiped, and
\[
|T(Q_k)| = d |Q_k|.
\]
Since \( T(E) \subseteq U T(Q_k) \), we have

\[
|T(E)| \leq \sum_k |T(Q_k)|
\]

\[
= d \sum_k |Q_k|
\]

\[
\leq d |E| + d \varepsilon.
\]

As this is true \( \forall \varepsilon > 0 \), we have \( |T(E)| \leq d |E| \).

To obtain the converse inequality, let \( U \supseteq E \)
be an open set with \( |U \setminus E| < \varepsilon \). Then
we can write \( U = \bigcup_k Q_k \) as a countable
union of nonoverlapping cubes. Then

\[
|U| = \sum_k |Q_k|
\]

and since each \( T(Q_k) \) is a nonoverlapping parallelepiped,

\[
|T(u)| = \sum_k |T(Q_k)| = d \sum_k |Q_k| = d \cdot m(u).
\]

On the other hand,
\[ |T(u)| \leq |T(E)| + |T(u \setminus E)| \]
\[ \leq |T(E)| + d |u \setminus E| \text{ by previous case} \]
\[ < |T(E)| + d \varepsilon. \]

Here
\[ |T(E)| > |T(u)| - d \varepsilon \]
\[ = d |u| - d \varepsilon \]
\[ > d |E| - d \varepsilon. \]

Since \( \varepsilon \) is arbitrary, we obtain \( |T(E)| \geq d |E|. \]

Remark
We have assumed that the Lebesgue measure of a parallelepiped is equal to its usual volume. This does require proof, but can be shown in a similar manner to how we did it for cubes.

Corollary
Lebesgue measure is invariant under rotations.
Theorem: Linear Changes of Variable.

Let \( T: \mathbb{R}^n \to \mathbb{R}^n \) be linear, and let \( f: \mathbb{R}^n \to \mathbb{R} \) be \( C \).

a. \((f \circ T)(x) = f(Tx)\) is \( C \).

b. If \( f \geq 0 \) or if \( f \in L^1(\mathbb{R}^n) \), then
\[
\int f(Tx) \, dx = |\det(T)| \int f(x) \, dx
\]

Proof Sketch

Write \( T \) as a composition of finitely many:

- Coordinate scalings:
  \[
  T(x_1, \ldots, x_n) = (x_1, \ldots, cx_j, \ldots, x_n) \quad (\det(T) = c)
  \]

- Shears:
  \[
  T(x_1, \ldots, x_n) = (x_1, \ldots, x_j + cx_k, \ldots, x_n) \quad (\det(T) = 1)
  \]

- Coordinate interchanges:
  \[
  T(x_1, \ldots, x_n) = (x_1, \ldots, x_k, \ldots, x_j, \ldots, x_n) \quad (\det(T) = -1)
  \]

Apply Fubini/Tonelli to each of these, e.g.,
\[ \int \int f(x_1, x_2 + cx_1) \, dx_1 \, dx_2 \]
\[ = \int \left( \int f(x_1, x_2 + cx_1) \, dx_2 \right) \, dx_1 \]
\[ = \int \left( \int f(x_1, x_2) \, dx_2 \right) \, dx_1 \]

Etc. Trace through, total effect is multiplication by \( | \det(T) | \).

Alternative approach

If \( f = \chi_E \) then \( f \circ T = \chi_{T^{-1}(E)} \). Extend to simple functions & arbitrary functions.