

3.3 Complex Measures

Definition

A function $\nu: \mathcal{M} \rightarrow \mathbb{C}$ is a complex measure on a ~~measurable space~~ measurable space (X, \mathcal{M}) if

a. $\nu(\emptyset) = 0$

b. If $E_1, E_2, \dots \in \mathcal{M}$ are disjoint, then

$$\nu\left(\bigcup_k E_k\right) = \sum_k \nu(E_k).$$

Remark

As was the case for signed measures, since the union $\bigcup_k E_k$ is independent of order, the series $\sum_k \nu(E_k)$ must converge unconditionally, & therefore must converge absolutely, i.e., $\sum_k |\nu(E_k)|$.

Note

A complex measure can only take complex values, & hence all complex measures are finite measures by definition.

Furthermore, they are bounded measures in the following sense

Exercise/Notation

Let ν be a complex measure. Define

$$\nu_r(E) = \operatorname{Re} \nu(E) \quad \& \quad \nu_i(E) = \operatorname{Im} \nu(E), \quad E \in \mathcal{M}.$$

Show that ν_r, ν_i are bounded signed measures, & for $E \in \mathcal{M}$ we have

$$|\nu(E)| \leq |\nu_r(E)| + |\nu_i(E)| \leq |\nu_r|(X) + |\nu_i|(X)$$

Therefore ν is bounded in the sense that

$$\sup_{E \in \mathcal{M}} |\nu(E)| < \infty$$

We refer to ν_r as the real part of ν , and ν_i as the imaginary part.

Definition: Integration

If ν is a complex measure, then we set

$$L'(\nu) = L'(\nu_r) \cap L'(\nu_i) = L'(|\nu_r|) \cap L'(|\nu_i|).$$

If $f \in L'(\nu)$, then

$$\int f d\nu = \int f d\nu_r + i \int f d\nu_i$$

Note that $\int f d\nu$ is a well-defined complex scalar.

Most definitions ~~like~~ likewise carry over by applying them to the real & imaginary parts.

Definition

A complex measure ν on (X, \mathcal{M}) is absolutely continuous w.r.t. a positive measure μ on (X, \mathcal{M}) , denoted $\nu \ll \mu$, if

$$\forall E \in \mathcal{M}, \quad \mu(E) = 0 \Rightarrow \nu(E) = 0$$

Equivalently, $\nu \ll \mu$ if $\nu_r \ll \mu$ & $\nu_i \ll \mu$

Exercise

If μ is a positive measure & $g \in L^1(\mu)$ (complex version),

then $d\nu = g d\mu$ defines a complex measure & $\nu \ll \mu$.

Definition

A complex measure is singular w.r.t. another complex measure μ , denoted $\nu \perp \mu$, if

$$\nu_r \perp \mu_r, \quad \nu_r \perp \mu_i, \quad \nu_i \perp \mu_r, \quad \nu_i \perp \mu_i.$$

Lebesgue-Radon-Nikodym Theorem

Let ν be a complex measure on (X, \mathcal{M}) and μ a σ -finite positive measure on (X, \mathcal{M}) .

Then $\exists f \in L^1(\mu)$ & a complex measure λ s.t.

$$\nu = f d\mu + \lambda, \quad \lambda \perp \mu$$

If we also have $\nu = \tilde{f} d\mu + \tilde{\lambda}$ where $\tilde{f} \in L^1(\mu)$ and $\tilde{\lambda} \perp \mu$, then

$$\tilde{\lambda} = \lambda \quad \& \quad \tilde{f} = f \quad \mu\text{-a.e.}$$

We need the following exercise in order to define the total variation of a complex measure.

Exercise

Let ν be a complex measure on (X, \mathcal{M}) . Let

$$\mu = |\nu_r| + |\nu_i|,$$

and show that μ is a ^{bounded} positive measure with

$\nu \ll \mu$. Conclude that $d\nu = f d\mu$ for some $f \in L^1(\mu)$.

Remark

The total variation of a complex measure is a little more awkward to define than it is for a signed measure. By the preceding exercise, there is at least one positive measure μ & a function $f \in L^1(\mu)$ such that $d\nu = f d\mu$, and we define the total variation $\|\nu\|$ of ν to be the measure

$$d\|\nu\| = |f| d\mu.$$

However, we want this definition to be independent of the choice of μ & f , which is the content of the following theorem.

Theorem

Let ν be a complex measure on (X, \mathcal{M}) .

If μ_1, μ_2 are bounded positive measures & $f_1 \in L^1(\mu_1)$, $f_2 \in L^1(\mu_2)$ are such that

$$f_1 d\mu_1 = d\nu = f_2 d\mu_2,$$

then $|f_1| d\mu_1 = |f_2| d\mu_2$.

Proof:

Since $\mu_1 \ll \mu_1 + \mu_2$, $\exists g_1 \in L^1(\mu_1 + \mu_2)$ s.t.

$d\mu_1 = g_1 d(\mu_1 + \mu_2)$. Likewise $\exists g_2 \in L^1(\mu_1 + \mu_2)$ s.t.

$d\mu_2 = g_2 d(\mu_1 + \mu_2)$. Because μ_1, μ_2 , & $\mu_1 + \mu_2$

are all positive measures, we have $g_1, g_2 \geq 0$

$(\mu_1 + \mu_2)$ -a.e.

Now, by a previous exercise, since

$d\nu = f_1 d\mu_1$ & $d\mu_1 = g_1 d(\mu_1 + \mu_2)$, we have

$$d\nu = f_1 g_1 d(\mu_1 + \mu_2).$$

Likewise

$$d\nu = f_2 g_2 d(\mu_1 + \mu_2).$$

But by the uniqueness clause of the Lebesgue-Radon-Nikodym Theorem implies that

$$f_1 g_1 = f_2 g_2 \quad (\mu_1 + \mu_2) - \text{a.e.}$$

Consequently,

$$|f_1| g_1 = |f_1 g_1| = |f_2 g_2| = |f_2| g_2 \quad (\mu_1 + \mu_2) - \text{a.e.},$$

and hence

$$\begin{aligned} |f_1| d\mu_1 &= |f_1| g_1 d(\mu_1 + \mu_2) \\ &= |f_2| g_2 d(\mu_1 + \mu_2) \\ &= |f_2| d\mu_2. \quad \blacksquare \end{aligned}$$

Definition

Let ν be a complex measure on (X, \mathcal{M}) . Then the total variation $|\nu|$ of ν is the positive measure $d|\nu| = |f| d\mu$ where μ is any positive bounded measure and f any function in $L^1(\mu)$ such that $d\nu = f d\mu$.

Exercise

Let ν be a complex measure on (X, \mathcal{M}) .

a. Show $|\nu(E)| \leq |\nu|(E) \quad \forall E \in \mathcal{M}$.

b. Show $\nu \ll |\nu|$, & $\exists g$ with $|g|=1$ $|\nu|$ -a.e.

such that $d\nu = g d|\nu|$. (This is called the polar decomposition of ν .)

c. If $f \in L^1(\nu)$, then $|\int f d\nu| \leq \int |f| d|\nu|$.

The following exercise gives some equivalent reformulations of the total variation that are sometimes easier to apply than our definition. In many texts, one of these forms is taken as the definition of $|\nu|$, and the other forms are ~~then~~ derived from it.

Hints

a. By definition of total variation, $d|\nu| = |f| d\mu$ where $\mu \geq 0$, $f \in L^1(\mu)$, & $d\nu = f d\mu$. Hence $|\nu(E)| = |\int_E f d\mu|$.

b. Part a implies $\nu \ll |\nu|$. So $d\nu = g d|\nu|$ for some $g \in L^1(|\nu|)$. Then $d|\nu| = |g| d|\nu|$. Show $\int_E (|g|-1) d|\nu| = 0 \quad \forall E \in \mathcal{M}$. Use Q25 to show $|g|=1$ $|\nu|$ -a.e.

Exercise

If ν is a complex measure on (X, \mathcal{M}) , then:

$$a. |\nu|(E) = \sup \left\{ \sum_{k=1}^n |\nu(E_k)| : n \in \mathbb{N}, E_k \in \mathcal{M}, E = \bigcup_{k=1}^n E_k \text{ disjointly} \right\}$$

$$b. |\nu|(E) = \sup \left\{ \sum_{k=1}^{\infty} |\nu(E_k)| : E_k \in \mathcal{M}, E = \bigcup_{k=1}^{\infty} E_k \text{ disjointly} \right\}$$

$$c. |\nu|(E) = \sup \left\{ \left| \int_E f d\nu \right| : |f| \leq 1 \text{ } |\nu|\text{-a.e.} \right\}$$

Hints:

Let $\mu_1(E)$, $\mu_2(E)$, $\mu_3(E)$ denote the quantities defined on the right-hand side of a, b, c. Show that $\mu_1 \leq \mu_2 \leq \mu_3 = |\nu|$ & $\mu_3 \leq \mu_1$.

More hints:

$\mu_2 \leq \mu_3$: Suppose $E = \bigcup_{k=1}^{\infty} E_k$ disjointly. Let c_k be the scalars of unit modulus such that $|\int_{E_k} d\nu| = c_k \int_{E_k} d\nu$, and set $f = \sum_k c_k \chi_{E_k}$.

$|\nu| \leq \mu_3$: $\exists f$ with $|f|=1$ ν -a.e. such that $d\nu = f d|\nu|$. Then $\int f d\nu = \int f f d|\nu| = \int |f|^2 d|\nu| = \int d|\nu|$.

$\mu_3 \leq \mu_1$: Suppose $|f| \leq 1$. Since f is bounded,
 \exists simple functions that converge to f
 uniformly. Hence $\forall \epsilon > 0$, \exists
 $\phi = \sum_{j=1}^N c_j \chi_{E_j}$ such that

$$\sup_{x \in X} |f(x) - \phi(x)| \leq \epsilon. \quad \text{Consequently}$$

$$|c_j| \leq 1 + \epsilon \quad \text{since } c_j = \phi(x) \text{ for some } x,$$

$$\text{and } |f d\nu| \leq |\int \phi d\nu| + \epsilon \|\nu\|.$$

Exercise

Let ν be a complex measure. Show

$$E \in \mathcal{M} \text{ is a } \nu\text{-null set} \iff |\nu|(E) = 0.$$

Notation

For a complex measure ν , we say a property holds ν -a.e. if it holds ^{except} on a null set for ν . Hence ν -a.e. is the same as $|\nu|$ -a.e.

Exercise

Let

$$M_b(X) = \{ \nu : \nu \text{ is a complex measure on } (X, \mathcal{M}) \}.$$

a. Show that $M_b(X)$ is a (complex) vector space, and that

$$\|\nu\| = |\nu|(X)$$

is a norm on $M_b(X)$.

b. Show that $M_b(X)$ is a Banach space, i.e., it is complete w.r.t. the norm $\|\cdot\|$ (this is not trivial, don't worry if you don't get it completely).

Hint: Suppose $\{\nu_n\}_{n \in \mathbb{N}}$ is Cauchy in $M_b(X)$.

Fix $E \in \mathcal{M}$. Show $\{\nu_n(E)\}_{n \in \mathbb{N}}$ is a Cauchy

sequence of complex scalars. Define $\nu(E) = \lim_{n \rightarrow \infty} \nu_n(E)$.

Show ν is a complex measure. Hard part:

Show $\|\nu - \nu_n\| \rightarrow 0$.

Exercise

Let μ be a positive measure on (X, \mathcal{M}) . Show

that if $f \in L^1(\mu)$ & $d\nu = f d\mu$, then

$$\|\nu\| = \|f\|_1 = \int |f| d\mu.$$

Consequently, $f \mapsto f d\mu$ is an isometric embedding of $L^1(\mu)$ into $M_b(X)$.

Exercises

1. Show that if $f: X \rightarrow \mathbb{C}$ is \mathcal{M} , then \exists simple functions ϕ_k such that $\phi_k(x) \rightarrow f(x)$ pointwise, and $|\phi_k(x)| \leq |f(x)| \quad \forall x \in X, k \in \mathbb{N}$.
2. Let μ be a positive measure on (X, \mathcal{M}) , & fix $g \in L^1(\mu)$. Define $d\nu = g d\mu$. Show that if $f \in L^1(\nu)$, then $\int f d\nu = \int fg d\mu$.
3. Show that if ν is a complex measure on (X, \mathcal{M}) and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.
Hint: Consider the polar form of ν .
4. Let ν be a complex measure & μ a positive measure on (X, \mathcal{M}) . Show that

$$\nu \ll \mu \iff |\nu| \ll \mu.$$
5. Define the complex conjugate of a complex measure ν by

$$\bar{\nu}(E) = \overline{\nu(E)}, \quad E \in \mathcal{M}.$$

Show that $\bar{\nu}$ is a complex measure, and

$$\int f d\bar{\nu} = \overline{\int \bar{f} d\nu} \quad \text{for } f \in L^1(\nu).$$

6. Given a complex measure ν on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, find an explicit description of $|\nu|$.

7. Show that

$$\nu \mapsto (\nu\{k\})_{k \in \mathbb{N}}$$

is an isometric isomorphism of $M_b(\mathbb{N})$

onto $\ell^1(\mathbb{N})$. Thus $M_b(\mathbb{N}) \cong \ell^1(\mathbb{N})$.