3.4 Differentiation on Euclidean space

The Radon-Nikodym derivative provides us with a kind of abstract version of the derivative:

$$\nu(E) = \int_E f \, d\mu \quad \Rightarrow \quad f = \frac{d\nu}{d\mu}$$

In this section we specialize to the case of Lebesgue measure on $\mathbb{R}^d$ and obtain more refined results.

Motivation

Differentiating is a type of averaging. Suppose that f is continuous on $\mathbb{R}$ and set $d\nu = f \, dx$.

By the Fundamental Theorem of Calculus,

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt$$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt}{h}$$

$$= \frac{d}{dx} \int_a^x f(t) \, dt$$

$$= f(x) \quad (FTC)$$

Similarly,

$$\lim_{h\to 0} \frac{\nu(x-h,x+h)}{|(x-h,x+h)|} = \lim_{h\to 0} \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt = f(x).$$

We will explore the connection between the fact and the fact that f is the Radon-Nikodym derivative of ν w.r.t. Lebesgue measure: $f = \frac{d\nu}{dx}$. We will see that this connection holds under much more general conditions than just for continuous f, and in this sense generalizes the FTC.

We will need the following technical lemma.

Simple Vitali Lemma

Let $\mathcal{C}$ be a collection of open balls in $\mathbb{R}^n$.

Let $U = \bigcup_{B \in \mathcal{C}} B$, and fix any constant $0 < c < |U|$.

Then $\exists$ disjoint balls $B_1, \dots, B_k \in \mathcal{C}$ such that

$$\sum_{j=1}^{k} |B_j| > \frac{c}{3^n}.$$

Proof:

Since U is ⓜ & $c < |U|$, $\exists$ compact $K \subseteq U$ with

$$|K| > c.$$

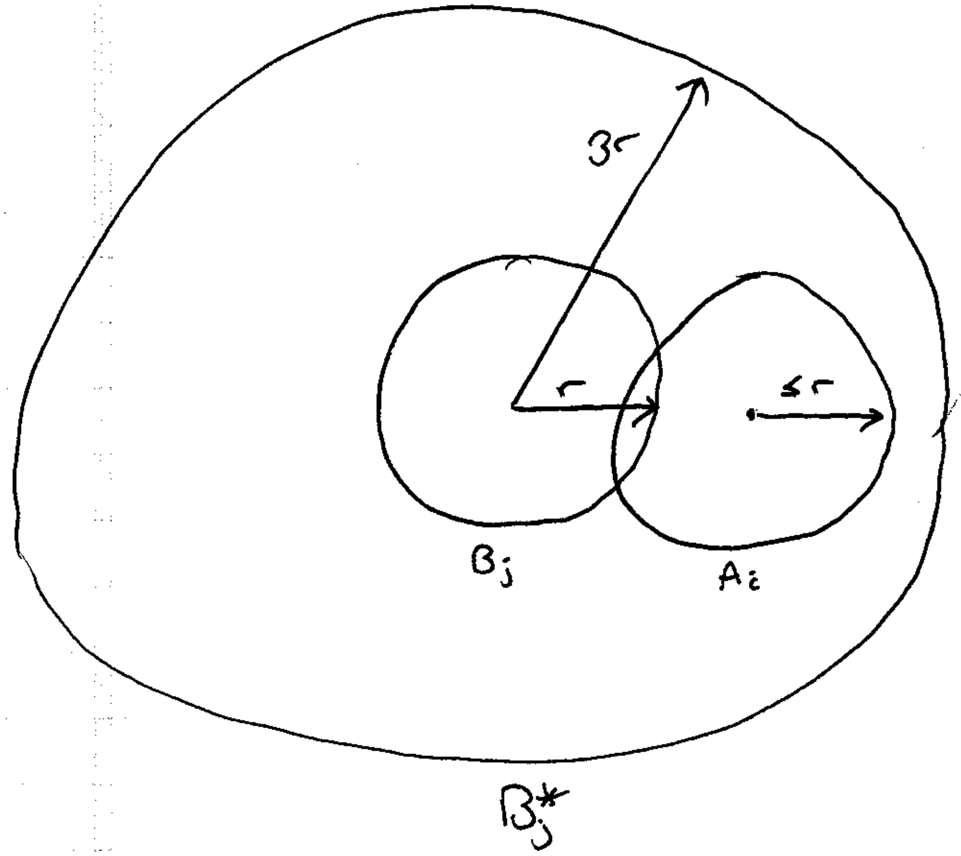
Since K is compact & $\mathcal{C}$ is an open cover of K, $\exists$ finitely many $A_1, \dots, A_m \in \mathcal{C}$ s.t.

$$K \subseteq \bigcup_{j=1}^{m} A_j.$$

Since the A_j are finitely many open balls, we can let B_1 be an A_j ball with maximal radius. If there are no A_j balls disjoint from B_1, then set $k=1$ and stop. Otherwise let B_2 be the largest A_j ball that is disjoint from B_1. Repeat this process, which must eventually stop, to select disjoint balls $B_1, \ldots, B_k$ from $A_1, \ldots, A_m$. The B_j balls need not cover K any more, but we hope they will cover an appropriate portion of K.

Let B_j^* denote the open ball that has the same center as B_j and 3 times larger radius. Suppose that A_i is not one of $B_1, \ldots, B_k$. Then A_i must intersect at least one of B_j, and let j be the smallest index for which that is true. Then

$$\text{radius}(A_i) \leq \text{radius}(B_j)$$



so we must have $A_i \subseteq B_j^*$. Thus

$$K \subseteq \bigcup_{i=1}^{m} A_i \subseteq \bigcup_{j=1}^{k} B_j^* .$$

Hence

$$c < |K| \leq \sum_{j=1}^{k} |B_j^*| = 3^n \sum_{j=1}^{n} |B_j| . \blacksquare$$

Remark

Although $L^1_{loc}(\mathbb{R})$ is a vector space, it is not a normed space. Instead, it's a good example of a topological vector space defined by a family of seminorms. These seminorms are

$$\|f\|_{1,K} = \|f\cdot\chi_K\|_1 = \int_K |f|, \quad K \text{ compact}$$

(or by using $B_k = \{x \in \mathbb{R}^n : \|x\| \le k\}$ we need only countably many seminorms). The topology on $L^1_{loc}(\mathbb{R}^n)$, & hence the meaning of convergence in $L^1_{loc}(\mathbb{R}^n)$, is determined by this family of seminorms.

Exercise

Show that $|x|^{\alpha} \in L^1_{loc}(\mathbb{R})$ for $\alpha > -1$,

but $|x|^{-1} \notin L^1_{loc}(\mathbb{R})$.

The analogue for $\mathbb{R}^n$ is $|x|^{\alpha} \in L^1_{loc}(\mathbb{R}^n)$ for $\alpha > -n$,
$|x|^{-n} \notin L^1_{loc}(\mathbb{R})$.

Definition

A Lebesgue measurable function $f: \mathbb{R}^n \longrightarrow \mathbb{C}$ is

locally integrable if

$$\forall \text{ compact } K \subseteq \mathbb{R}^n, \quad \int_K |f(x)|\, dx < \infty.$$

Exercise

If we set $B_k = \{x \in \mathbb{R}^n : \|x\| \le k\}$, then

f is locally integrably if & only if

$$\forall k \in \mathbb{N}, \quad \int_{\|x\| \le k} |f(x)|\, dx < \infty.$$

Example

$f(x) = x^2$ is locally integrable on $\mathbb{R}$, although

it is not integrable on $\mathbb{R}$.

Definition

$$L^1_{loc}(\mathbb{R}^n) = \{f : f \text{ is locally integrable on } \mathbb{R}^n\}.$$

Exercise

Show $L^1_{loc}(\mathbb{R}^n)$ is a (complex) vector space.

Definition: Averages

If $f \in L^1_{loc}(\mathbb{R}^n)$ and $r > 0$, then the average of f over the ball $B_r(x)$ of radius r centered at $x \in \mathbb{R}^n$ is

$$A_r f(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(t)\, dt.$$

Remark

$|B_r(x)| = C_n r^n$ where C_n depends only on n.

Lemma

If $f \in L^1_{loc}(\mathbb{R}^n)$, then $A_r f(x)$ is jointly continuous in both x & r, i.e.,

$$x_k \to x \;\; \& \;\; r_k \to r \;\Rightarrow\; A_{r_k} f(x_k) \to A_r f(x)$$

Proof (sketch)

We have

$$\chi_{B_{r_k}(x_k)} \longrightarrow \chi_{B_r(x)} \text{ pointwise a.e. as } n \to \infty.$$

Also, if $r_k < r + \frac{1}{2}$ & $\|x_k - x\| < \frac{1}{2}$ then

$$|\chi_{B_{r_k}(x_k)}| \leq \chi_{B_{r+1}(x)} \in L^1(\mathbb{R}^n).$$

The Dominated Convergence Theorem therefore implies that

$$\int_{B_{r_k}(x_k)} f \longrightarrow \int_{B_r(x)} f$$

Since $r_k \to r$ we therefore have

$$\begin{aligned} A_{r_k} f(x_k) &= \frac{1}{|B_{r_k}(x_k)|} \int_{B_{r_k}(x_k)} f \\ &= \frac{1}{C_n\, r_k^n} \int_{B_{r_k}(x_k)} f \\ &\longrightarrow \frac{1}{C_n\, r^n} \int_{B_r(x)} f \qquad \text{as } k \to \infty \\ &= \frac{1}{|B_r(x)|} \int_{B_r(x)} f \\ &= A_r f(x). \quad \blacksquare \end{aligned}$$

Exercise

Let $f, g \in L^1(\mathbb{R}^n)$ be given. Use Fubini's Theorem to show that

$$(f * g)(x) = \int f(y)\, g(x-y)\, dy$$

exists for a.e. x, that $f*g \in L^1(\mathbb{R}^n)$, and that

$$\|f*g\|_1 \leq \|f\|_1 \, \|g\|_1.$$

As a consequence, $L^1(\mathbb{R}^n)$ is a Banach algebra w.r.t. the convolution operation $*$.

Exercise

Let $\chi_r = \frac{1}{|B_r(0)|} \chi_{B_r(0)}$.

Show that

$$A_r f(x) = (f * \chi_r)(x)$$

Thus averaging is just a particular form of convolution. Or, from another viewpoint, general convolutions $f*g$ are a generalized form of averaging, where g is allowed to "weight" $\mathbb{R}^n$ differently that χ_r does.

<u>Definition</u>

If $f \in L^1_{loc}(\mathbb{R}^n)$, then its <u>Hardy–Littlewood maximal function</u> is

$$f^*(x) = Hf(x) = \sup_{r>0} A_r(|f|)(x)$$

$$= \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f|$$

$$= \sup_{r>0} \left(|f| * \chi_r \right)(x)$$

<u>Note</u>

Since

$$(Hf)^{-1}(a,\infty) = \bigcup_{r>0} (A_r|f|)^{-1}(a,\infty)$$

open since $A_r|f|$ is continuous

we conclude that $\{Hf > a\}$ is open & hence (m).

In fact, this shows that Hf is a continuous function on $\mathbb{R}^n$. This is not too surprising, since averaging is a type of smoothing process.

Remark

Compare the following theorem to Tchebyshev's inequality, which states that if $f \in L^1(\mathbb{R}^n)$, then

$$|\{|f| > \alpha\}| \leq \frac{1}{\alpha}\int |f|.$$

The Maximal Theorem

If $f \in L^1(\mathbb{R}^n)$, then

$$\forall \alpha > 0, \quad |\{Hf > \alpha\}| \leq \frac{3^n}{\alpha}\int |f|$$

Note that here we have $|f|$ and not just Hf

Proof:

Let

$$E_\alpha = \{Hf > \alpha\}.$$

If $x \in E_\alpha$ then

$$\alpha < Hf(x) = \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f|.$$

Hence $\exists\, r_x$ such that

$$\frac{1}{|B_{r_x}(x)|}\int_{B_{r_x}(x)} |f| > \alpha.$$

Note that $\bigcup_{x \in E_\alpha} B_{r_x}(x) \supseteq E_\alpha$. Therefore, if we fix a constant $0 < c < |E_\alpha|$, then the Simple Vitali Lemma implies that there exist finitely many $x_1, \ldots, x_k \in E_\alpha$ whose corresponding balls $B_{r_{x_k}}(x_k)$ are disjoint and satisfy

$$\sum_{j=1}^{k} |B_{r_{x_j}}(x_j)| > \frac{c}{3^n}.$$

Therefore

$$c < 3^n \sum_{j=1}^{k} |B_{r_{x_j}}(x_j)|$$

$$< 3^n \cdot \frac{1}{\alpha} \int_{B_{r_{x_j}}(x_j)} |f| \qquad \text{as } x_j \in E_\alpha$$

$$\leq \frac{3^n}{\alpha} \int |f|.$$

As this is true for all $c < |E_\alpha|$, we conclude that

$$|E_\alpha| \leq \frac{3^n}{\alpha} \int |f|. \quad \blacksquare$$

Weak L^1

A useful space that sometimes substitutes for L^1 in certain theorems where L^1 is not appropriate is the space Weak L^1.

A function f on $\mathbb{R}^n$ belongs to Weak L^1 if

$$\exists C>0 \ \forall \alpha>0, \quad |\{|f|>\alpha\}| \leq \frac{C}{\alpha}.$$

Exercise

Use Tchebyshev's inequality to show that [redacted].

$$L^1(\mathbb{R}^n) \subseteq \text{Weak } L^1(\mathbb{R}^n).$$

Find a function in Weak L^1 that is not integrable.
Hint: $|x|^{-n}$

Remark

The Maximal Theorem implies that

$$f \in L^1(\mathbb{R}^n) \implies Hf \in \text{Weak } L^1(\mathbb{R}^n).$$

Motivation

As in many theorems in analysis, the key to the proof of the next main result is to find a good approximation. In particular, we will need to approximate an L^1 function by a continuous function. So, we first take a short aside to show that this is always possible.

Definition

The support of a function $f: \mathbb{R}^n \to \bar{\mathbb{R}}$ or $\mathbb{C}$ is the closure of the set where f is nonzero:

$$\operatorname{supp}(f) = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$$

By definition, $\operatorname{supp}(f)$ is a closed set. Hence

$$f \text{ has compact support} \iff f = 0 \text{ outside a bounded set.}$$

Definition

$$C(\mathbb{R}) = \{ f : \mathbb{R}^n \to \mathbb{C} \; : \; f \text{ is continuous} \}$$

$$C_b(\mathbb{R}) = \{ f \in C(\mathbb{R}) : f \text{ is bounded} \}$$

$$C_c(\mathbb{R}) = \{ f \in C(\mathbb{R}) : \operatorname{supp}(f) \text{ is compact} \}.$$

Remark

Every function f in $C_c(\mathbb{R}^n)$ is uniformly continuous, i.e.,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad \forall x, y \in \mathbb{R}^n,$$

$$\|x - y\| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

The important point is that δ does not depend on x, this is what is "uniform" about uniform continuity.

Theorem

$C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.

That is, if $f \in L^1(\mathbb{R}^n)$, then $\exists\, g_k \in C_c(\mathbb{R}^n)$ s.t.

$$\|f - g_k\|_1 \longrightarrow 0.$$

or, equivalently,

$$\forall \varepsilon > 0 \quad \exists\, g \in C_c(\mathbb{R}^n) \quad \text{s.t.} \quad \|f - g\|_1 < \varepsilon.$$

Proof:

We begin with the case of integrable characteristic functions. That is, suppose that $f = \chi_E$ where

$E \subseteq \mathbb{R}^n$ is Lebesgue measurable and $|E| < \infty$ (so that $\chi_E \in L^1(\mathbb{R}^n)$).

Choose $\varepsilon > 0$. Then $\exists$ open $U \supseteq E$ with $|U \setminus E| < \varepsilon$ (so, in particular, $|U| < \infty$ as well). Then we can write $U = \bigcup_{k=1}^{\infty} Q_k$ as a union of nonoverlapping cubes. Applying the disjointization trick to the Q_k, we can write $U = \bigcup_{k=1}^{\infty} F_k$ disjointly, where each F_k is a "partly open" cube. Specifically, F_k is Q_k with part or all of its boundary removed, so $|F_k| = |Q_k|$.

Now,

$$|U| = \sum_{k=1}^{\infty} |F_k|.$$

Therefore, if we set

$$f_N = \sum_{k=1}^{N} \chi_{F_k} = \chi_{\bigcup_{k=1}^{N} F_k},$$

Then (exercise) $f_N \longrightarrow \chi_U$ in L^1-norm.

Hence $\exists N$ s.t.

$$\| \chi_U - f_N \|_1 < \varepsilon.$$

Now, f_N is not continuous, but it is a step function. Exercise: $\exists\, g_N \in C_c(\mathbb{R})$ with

$$\| f_N - g_N \|_1 < \varepsilon.$$

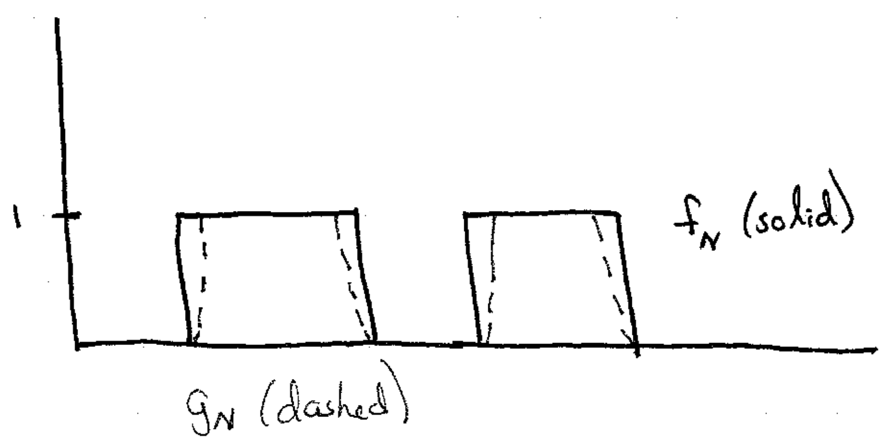


Illustration for $n = 1$

Consequently, by the triangle inequality, we have that

$$\begin{aligned}\|\chi_E - g_N\|_1 &= \|\chi_E - \chi_u + \chi_u - g_N + g_N - f_N\|_1 \\ &\le \|\chi_E - \chi_u\|_1 + \|\chi_u - g_N\|_1 + \|g_N - f_N\|_1 \\ &\le \varepsilon + \varepsilon + \varepsilon \\ &= 3\varepsilon.\end{aligned}$$

This proves the result for the case where f is a characteristic function.

Exercise: Extend to integrably simple functions $\phi = \sum_{j=1}^{N} a_j \chi_{E_j}$.

Finally, let $f \in L^1(\mathbb{R}^n)$ be arbitrary.

Exercise: Show $\exists$ simple functions ϕ_k such that $\phi_k \longrightarrow f$ in $L^1(\mathbb{R}^n)$.

Hint: If $f \geq 0$, then $\exists\, \phi_k \nearrow f$, & hence by the DCT, $\|f - \phi_k\|_1 \to 0$. Extend to arbitrary f by considering the positive & negative parts of the real & imaginary parts of f.

Consequently, given $\varepsilon > 0$, $\exists$ simple, integrable ϕ such that $\|f - \phi\|_1 < \varepsilon$. By the above exercise, $\exists\, g \in C_c(\mathbb{R}^n)$ s.t. $\|\phi - g\|_1 < \varepsilon$. Hence

$$\|f-g\|_1 \le \|f-\phi\|_1 + \|\phi - g\|_1$$
$$< \varepsilon + \varepsilon = 2\varepsilon.$$

This proves the general case. ∎

Remark

It is actually true that

$$C_c^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ is infinitely differentiable } \& \text{ compactly supported}\}$$

is dense in $L^1(\mathbb{R}^n)$!

Exercise

Define $T_a f(x) = f(x-a)$, i.e., translate f by a.

Show that if $f \in L^1(\mathbb{R})$, then $\lim_{a\to 0} \|T_a f - f\|_1 = 0$.

Hint: Prove it first for $g \in C_c(\mathbb{R})$ by making use of the fact that g is uniformly continuous. Then write

$$\|T_a f - f\|_1 \le \|T_a f - T_a g\|_1 + \|T_a g - g\|_1 + \|g - f\|_1.$$

Motivation

Recall that by the Fundamental Theorem of Calculus, if f is continuous on $\mathbb{R}$ then

$$f(x) = \lim_{h\to 0} \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} \frac{1}{|B_h(x)|}\int_{B_h(x)} f(t)\,dt$$

$$= \lim_{h\to 0} A_h f(x).$$

The following result will generalize this to functions in $L^1_{loc}(\mathbb{R})$, & show that

$$f(x) = \lim_{r\to 0} A_r f(x) \quad \text{a.e.} \quad \text{if} \quad f \in L^1_{loc}(\mathbb{R}^n).$$

Note the connection to the maximal function:

$$\lim_{r\to 0} A_r f(x) = \lim_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} f(t)\,dt$$

while

$$Hf(x) = \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(t)|\,dt$$

<u>Theorem</u> (Lebesgue Differentiation Theorem, first version)
If $f \in L^1_{loc}(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$,

$$f(x) = \lim_{r\to 0} A_r f(x)$$
$$= \lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(t)\, dt.$$

<u>Proof:</u>
For each $N \in \mathbb{N}$, define

$$f_N(x) = f(x)\chi_{B_N(0)}(x) = \begin{cases} f(x), & |x| < N, \\ 0, & |x| \geq N. \end{cases}$$

If we show that

$$\forall N, \qquad f_N(x) = \lim_{r\to 0} A_r f_N(x) \quad \text{a.e.},$$

then, since $f(x) = f_N(x)$ for N large enough, the result follows for f itself. Thus, we can replace f by the integrable functions f_N. Or, in other words, it suffices to prove the result for integrable functions.

Therefore, assume that $f \in L^1(\mathbb{R}^n)$.

By a previous theorem, given $\varepsilon > 0$, we can find $g \in C_c(\mathbb{R}^n)$ such that

$$\|f - g\|_1 < \varepsilon.$$

Now, since g is uniformly continous, if we choose $\eta > 0$ then $\exists\, r > 0$ such that

$$\|x - y\| < r \Rightarrow |g(x) - g(y)| < \eta.$$

Hence

$$|A_r g(x) - g(x)| = \left| \frac{1}{|B_r(x)|} \int_{B_r(x)} g(y)\, dy - g(x) \right|$$

$$= \left| \frac{1}{|B_r(x)|} \int_{B_r(x)} g(y)\, dy - g(x) \frac{1}{|B_r(x)|} \int_{B_r(x)} dy \right|$$

$$= \frac{1}{|B_r(x)|} \left| \int_{B_r(x)} \big(g(y) - g(x) \big)\, dy \right|$$

$$\leq \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y) - g(x)|\, dy$$

$$\leq \frac{1}{|B_r(x)|} \int_{B_r(x)} \eta\, dy$$

$$= \eta.$$

In fact, the same argument shows that if $s \le r$ then

$$|A_s g(x) - g(x)| \le \eta \quad \forall x \in \mathbb{R}^n.$$

Hence

$$\forall \eta > 0 \ \exists r > 0 \text{ s.t. } \forall s \le r \ \forall x \in \mathbb{R}^n,$$

$$|A_s g(x) - g(x)| \le \eta.$$

This says that $A_r g(x) \longrightarrow g(x)$ <u>uniformly</u> as $r \to 0$.

Using the continuous-domain version of limsup, we therefore have $\forall x \in \mathbb{R}^n$ that

$$\limsup_{r\to 0} |A_r f(x) - f(x)|$$

$$= \limsup_{r\to 0} \Big| A_r f(x) - A_r g(x) + A_r g(x) - g(x) + g(x) - f(x) \Big|$$

$$\le \limsup_{r\to 0} |A_r (f-g)(x)| + \limsup_{r\to 0} |A_r g(x) - g(x)|$$

$$+ \limsup_{r\to 0} |g(x) - f(x)|$$

$$\le H(f-g)(x) + 0 + |g(x) - f(x)|. \quad (*)$$

Now, our goal is to show that $\limsup_{r\to 0} |A_r f(x) - f(x)| = 0$ a.e.

To this end, consider the set

$$E_\alpha = \left\{ \limsup_{r\to 0} |A_r f - f| > \alpha \right\}$$

From equation (*) if $x \in E_\alpha$ then we must have either $H(f-g)(x) > \frac{\alpha}{2}$ or $|g(x) - f(x)| > \frac{\alpha}{2}$ (or both). Hence

$$E_\alpha \subseteq F_{\alpha/2} \cup G_{\alpha/2}$$

where

$$F_{\alpha/2} = \left\{ H(f-g) > \frac{\alpha}{2} \right\},$$

$$G_{\alpha/2} = \left\{ |g-f| > \frac{\alpha}{2} \right\}.$$

Now, by the Maximal Theorem,

$$|F_{\alpha/2}| \leq \frac{3^n}{\alpha/2} \int |f-g|$$

$$= \frac{2\cdot 3^n}{\alpha} \|f-g\|_1$$

$$< \frac{2\cdot 3^n \varepsilon}{\alpha} = \frac{C\varepsilon}{\alpha},$$

where $C = 2\cdot 3^n$ is a fixed constant.

Also, by Tchebyshev's Inequality,

$$\begin{aligned} |G_{\alpha/2}| &= \left| \{ |g-f| > \tfrac{\alpha}{2} \} \right| \\ &\le \frac{1}{\alpha/2} \int |f-g| \\ &= \frac{2}{\alpha} \|f-g\|_1 \\ &< \frac{2\varepsilon}{\alpha}. \end{aligned}$$

Hence

$$\begin{aligned} |E_\alpha| &\le |F_{\alpha/2}| + |G_{\alpha/2}| \\ &\le \frac{C\varepsilon}{\alpha} + \frac{2\varepsilon}{\alpha} \\ &= \left(\frac{C+2}{\alpha}\right)\varepsilon. \end{aligned}$$

Since ε is arbitrary, we conclude that $|E_\alpha| = 0$.

Finally,

$$\{ \limsup_{r\to 0} |A_r f - f| > 0 \} = \bigcup_{n=1}^{\infty} E_{1/n},$$

so $\limsup_{r\to 0} |A_r f - f| = 0$ a.e. ∎

<u>Improvements</u>

We've shown that if $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$f(x) = \lim_{r \to 0} A_r f(x) = \lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)\, dy$$

and consequently

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \big(f(x) - f(y) \big)\, dy = 0. \quad (*).$$

We will obtain some improvements to this result.

<u>Definition</u>
The <u>Lebesgue set</u> of $f \in L^1_{loc}(\mathbb{R}^n)$ is

$$L_f = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x) - f(y)|\, dy = 0 \right\}.$$

Note that this is a little stronger requirement than what appears in $(*)$.

<u>Exercise</u>
If f is continuous at x, then $x \in L_f$.

Notation

The points x in L_f are the Lebesgue points of f.

Thus, every point of continuity is a Lebesgue point, but the converse need not be true.

Theorem

If $f \in L^1_{loc}(\mathbb{R}^n)$ then the Lebesgue set of f is almost all of $\mathbb{R}^n$, i.e.,

$$|L_f^c| = 0.$$

Proof:

Given $c \in \mathbb{C}$, define $g_c(x) = |f(x) - c|$. Then $g_c \in L^1_{loc}(\mathbb{R}^n)$, so by the preceding theorem we have

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} g_c(y)\, dy = g_c(x) \quad \text{a.e.},$$

or,

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - c|\, dy = |f(x) - c| \quad \text{a.e.} \quad (*)$$

Let E_c be the set of measure zero where $(*)$ does not hold.

Let D be a countable & dense subset of $\mathbb{C}$, for example, $D = \mathbb{Q} + i\mathbb{Q}$. Let

$$E = \bigcup_{c \in D} E_c,$$

so $|E| = 0$ since D is countable.

Suppose $x \notin E$, & choose $\varepsilon > 0$. Since $f(x) \in \mathbb{C}$ and D is dense in $\mathbb{C}$, we can find a $c \in D$ such that

$$|f(x) - c| < \varepsilon.$$

Since $x \notin E$ we have $x \notin E_c$, so

$$\limsup_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x) - f(y)| \, dy$$

$$\leq \limsup_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \Big(|f(x) - c| + |c - f(y)| \Big) \, dy$$

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$$\le \limsup_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x) - c| \, dy$$

$$+ \limsup_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - c| \, dy$$

$$= \underbrace{|f(x) - c|}_{\text{integrating a constant}} + \underbrace{|f(x) - c|}_{\text{since } x \notin E_c}$$

$$< \varepsilon + \varepsilon = 2\varepsilon.$$

Since ε is arbitrary, we conclude that

$$0 \le \limsup_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x) - f(y)| \, dy \le 0,$$

and hence x is a Lebesgue point. Thus $E^c \subseteq L_f$, so $L_f^c \subseteq E$, which has measure zero. ∎

Regularly Shrinking Sets

Actually, in our previous discussion there is nothing special about averaging over balls. We can average over cubes or any other sequence of sets that "shrinks nicely" to x, in the following sense.

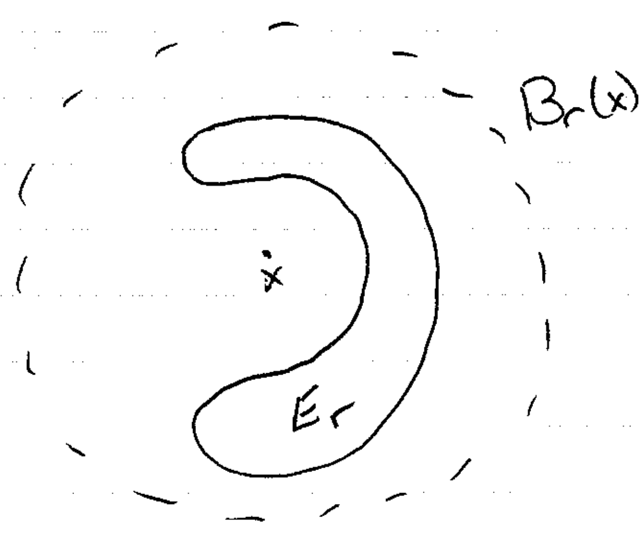
Definition

A family $\{E_r\}_{r>0}$ shrinks regularly to $x \in \mathbb{R}^n$ if

a. $E_r \subseteq B_r(x) \quad \forall r > 0$

b. $\exists \alpha > 0$ s.t. $\forall r > 0, \quad |E_r| > \alpha \, |B_r(x)|$

Note that E_r need not contain x, it only needs to be contained in the ball $B_r(x)$ & have at least a minimum percentage of the measure of the ball.



The Lebesgue Differentiation Theorem

Given $f \in L^1_{loc}(\mathbb{R}^n)$, let x be any Lebesgue point of f (this includes a.e. point in $\mathbb{R}^n$).

If $\{E_r\}_{r>0}$ is any family of sets that shrinks regularly to x, then

$$\lim_{r \to 0} \frac{1}{|E_r|} \int_{E_r} |f(y) - f(x)| \, dy = 0$$

and

$$\lim_{r \to 0} \frac{1}{|E_r|} \int_{E_r} f(y) \, dy = f(x).$$

Proof

With $\alpha > 0$ being the number appearing in the definition of shrinking regularly, we have

$$\frac{1}{|E_r|} \int_{E_r} |f(y) - f(x)| \, dy$$

$$\leq \frac{1}{\alpha |B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| \, dy$$

since $|E_r| > \alpha |B_r(x)|$ & $E_r \subseteq B_r(x)$

$$\longrightarrow 0 \text{ as } r \to 0. \quad \blacksquare$$

Example

For $n=1$ the Lebesgue differentiation Theorem tells us that if $f \in L^1_{loc}(\mathbb{R})$ then

$$\lim_{h\to 0} \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, dy = f(x) \quad \text{a.e.}$$

This is because balls are intervals: $B_r(x) = (x-r, x+r)$.

However, since $\{[x, x+h)\}_{h>0}$ is a regularly shrinking family, we also get

$$\lim_{h\to 0} \frac{1}{h} \int_{x}^{x+h} f(y)\, dy = f(x) \quad \text{a.e.}$$

Regularity of Borel Measures

Definition

We will say that a positive Borel measure ν on $\mathbb{R}^n$ is *regular* if:

a. it is locally finite, ie., $\nu(K) < \infty$ for every compact $K \subseteq \mathbb{R}^n$, and

b. $\forall$ Borel set $E \subseteq \mathbb{R}^n$,

$$\nu(E) = \inf\{\nu(U) : \text{open } U \supseteq E\}.$$

Remark: It requires proof, but condition b actually is a consequence of condition a.

Exercise

$$\text{locally finite} \Rightarrow \sigma\text{-finite}$$

but

$$\sigma\text{-finite} \not\Rightarrow \text{locally finite}.$$

Definition

A signed or complex Borel measure ν is said to be regular if $|\nu|$ is regular.

Exercise

Show that if $f \in L^1_{loc}(\mathbb{R}^n)$ and $f \geq 0$, then $d\nu = f\,dx$ is regular.

Theorem

Let ν be a regular signed or complex Borel measure on $\mathbb{R}^n$. Let

$$d\nu = f\,dx + d\lambda$$

be the Lebesgue-Radon-Nikodym decomposition of ν w.r.t. Lebesgue measure dx.

Then for (Lebesgue) a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} \frac{\nu(E_r)}{|E_r|} = f(x)$$

for every family $\{E_r\}_{r>0}$ that shrinks regularly to x. In particular,

$$\lim_{r \to 0} \frac{\nu(B_r(x))}{|B_r(x)|} = f(x) \text{ a.e.}$$

Proof:

Exercise: $\quad d|\nu| = |f|\,dx + d|\lambda|$.

Exercise: The regularity of $|\nu|$ implies that both

$|f|\,dx$ and $|\lambda|$ are regular as well.

Assume first that $E_r = B_r(x)$ and that $\lambda \geq 0$. Since $\lambda \perp dx$, $\exists$ a Borel set $A \subseteq \mathbb{R}$ such that

$$\lambda(A) = 0 = |A^c|.$$

Set

$$F_k = \left\{ x \in A : \limsup_{r \to 0} \frac{\lambda(B_r(x))}{|B_r(x)|} > \frac{1}{k} \right\}$$

We will show that $|F_k| = 0$ for each $k \in \mathbb{N}$.

Now, since λ is regular, given $\varepsilon > 0$ we can find an open $U_\varepsilon \supseteq A$ such that $\lambda(U_\varepsilon \setminus A) < \varepsilon$. Since $\lambda(A) = 0$, we therefore have

$$\lambda(U_\varepsilon) < \varepsilon.$$

Now, if $x \in F_k$ is fixed then by definition of limsup we can find a sequence of radii $r_n \to 0$ such that

$$(*) \qquad \frac{\lambda(B_{r_n}(x))}{|B_{r_n}(x)|} > \frac{1}{k} \quad \forall n \in \mathbb{N}.$$

Since $x \in F_k \subseteq U_\varepsilon$, if we choose n large enough then we will also have

$$(**) \qquad B_{r_n}(x) \subseteq U_\varepsilon.$$

Fix any n large enough that both $(*)$ & $(**)$ hold, and define

$$B_x = B_{r_n}(x).$$

Now let

$$V_\varepsilon = \bigcup_{x \in F_k} B_x.$$

Fix any $0 < c < |V_\varepsilon|$. Then by the Simple Vitali Lemma, $\exists\, x_1, \dots, x_N \in F_k$ such that the corresponding balls $B_{x_1}, \dots, B_{x_N}$ are <u>disjoint</u> and satisfy

$$\sum_{j=1}^{N} |B_{x_j}| > \frac{c}{3^n}.$$

Consequently,

$$\begin{aligned} c &< 3^n \sum_{j=1}^{N} |B_{x_j}| \\ &< 3^n k \sum_{j=1}^{N} \lambda(B_{x_j}) \\ &\le 3^n k \, \lambda(U_\varepsilon) \qquad \text{since } B_{x_j} \subseteq U_\varepsilon \text{ \& are disjoint} \\ &< 3^n k \, \varepsilon. \end{aligned}$$

As n, k are fixed & $0 < c < |V_\varepsilon|$ is arbitrary, we conclude $|V_\varepsilon| \le 3^n k \varepsilon$. But ε is also arbitrary so $|V_\varepsilon| = 0$ & hence $|F_k| = 0$ as $F_k \subseteq V_\varepsilon$.

Hence $F = \bigcup_{k \in \mathbb{N}} F_k$ also has measure zero, so the proof is complete for the case where $E_r = B_r(x)$. and $\lambda \geqslant 0$. Exercise: Extend to the general case. ∎