REAL ANALYSIS LECTURE NOTES:

3.5 ABSOLUTELY CONTINUOUS AND SINGULAR FUNCTIONS

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In these notes we will expand on the second part of Section 3.5 of Folland's text, covering the properties of absolutely continuous functions on the real line (which are those functions for which the Fundamental Theorem of Calculus holds) and singular functions on $\mathbb{R}$ (which are differentiable at almost every point but have the property that the derivative is zero a.e.).

3.5.1 Singular Functions on $\mathbb{R}$

We begin with an example of a singular function.

Exercise 1 (The Cantor–Lebesgue Function)**.** Consider the two functions φ_1, φ_2 pictured in Figure 1. The function φ_1 takes the constant value $1/2$ on the interval $(1/3, 2/3)$ that is removed in the first stage of the construction of the Cantor middle-thirds set, and is linear on the remaining intervals. The function φ_2 also takes the same constant $1/2$ on the interval $(1/3, 2/3)$ but additionally is constant with values $1/4$ and $3/4$ on the two intervals that are removed in the second stage of the construction of the Cantor set. Continue this process, defining $\varphi_3, \varphi_4, \dots$, and prove the following facts.

(a) Each φ_k is monotone increasing on $[0, 1]$.

(b) $|\varphi_{k+1}(x) - \varphi_k(x)| < 2^{-k}$ for every $x \in [0, 1]$.

(c) $\varphi(x) = \lim_{k\to\infty} \varphi_k(x)$ converges uniformly on $[0, 1]$.

The function φ constructed in this manner is called the *Cantor–Lebesgue function* or, more picturesquely, the *Devil's staircase*. Prove the following facts about φ.

(d) φ is continuous and monotone increasing on $[0, 1]$, but φ is not uniformly continuous.

(e) φ is differentiable for a.e. $x \in [0, 1]$, and $\varphi'(x) = 0$ a.e.

(f) The Fundamental Theorem of Calculus does not apply to φ:

$$\varphi(1) - \varphi(0) \neq \int_0^1 \varphi'(x)\, dx.$$

If we extend φ to $\mathbb{R}$ by reflecting it about the point $x = 1$, and then extend by zero outside of $[0, 2]$, we obtain the continuous function φ pictured in Figure 2. It is interesting

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These notes follow and expand on the text "Real Analysis: Modern Techniques and their Applications," 2nd ed., by G. Folland. Additional material is based on the text "Measure and Integral," by R. L. Wheeden and A. Zygmund.

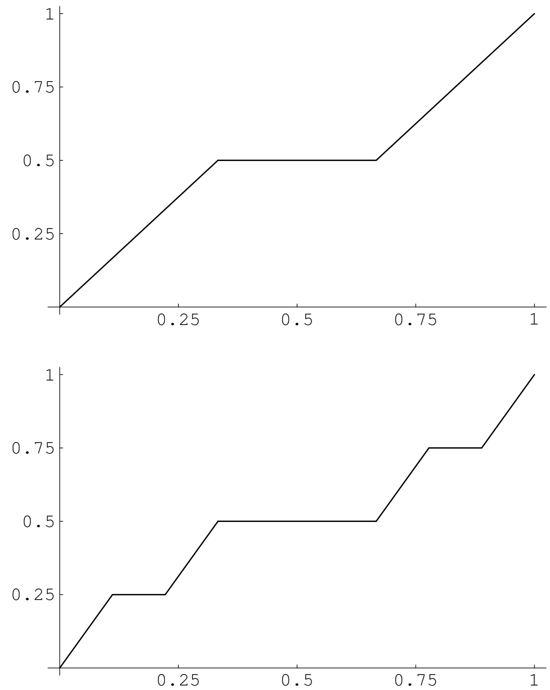

FIGURE 1. First stages in the construction of the Cantor–Lebesgue function.

that it can be shown that φ is an example of a *refinable function*, as it satisfies the following *refinement equation*:

$$\varphi(x) \;=\; \frac{1}{2}\varphi(3x) + \frac{1}{2}\varphi(3x-1) + \varphi(3x-2) + \frac{1}{2}\varphi(3x-3) + \frac{1}{2}\varphi(3x-4). \qquad (1)$$

Thus φ equals a finite linear combination of compressed and translated copies of itself, and so exhibits a type of self-similarity. Refinable functions are widely studied and play important roles in wavelet theory and in subdivision schemes in computer-aided graphics.

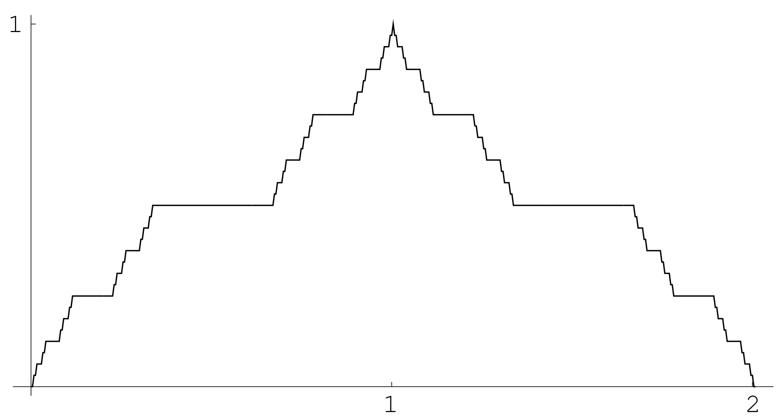

FIGURE 2. The reflected Devil's staircase (Cantor–Lebesgue function).

Exercise 2. The fact that φ is refinable yields easy recursive algorithms for plotting φ to any desired level of accuracy. For example, since we know the values of $\varphi(k)$ for k integer, we can compute the values $\varphi(k/3)$ for $k \in \mathbb{Z}$ by considering $x = k/3$ in equation (1). Iterating this, we can obtain the values $\varphi(k/3^j)$ for any $k \in \mathbb{Z}$, $j \in \mathbb{N}$. Plot the Cantor–Lebesgue function.

The Cantor–Lebesgue function is the prototypical example of a singular function.

Definition 3 (Singular Function)**.** A function $f\colon [a,b] \to \mathbb{C}$ or $f\colon \mathbb{R} \to \mathbb{C}$ is *singular* if f is differentiable at almost every point in its domain and $f' = 0$ a.e.

3.5.2 The Vitali Covering Lemma

Before proceeding further, we pause to prove a more refined version of the Simple Vitali Lemma that was used in Section 3.4. The version of theorem we give uses closed balls, but we could just as well use closed cubes or other appropriate families of sets.

Definition 4. Let $E \subseteq \mathbb{R}^d$ be given (E need not be Lebesgue measurable). Then a collection $\mathcal{B}$ of closed balls in $\mathbb{R}^d$ is a *Vitali cover* for E if

$$\forall\, x \in E, \quad \forall\, \eta > 0, \quad \exists\, B \in \mathcal{B} \text{ such that } x \in B \text{ and } \mathrm{radius}(B) < \eta.$$

We will need two exercises. The first one recalls one of the basic regularity properties of Lebesgue measure.

Exercise 5. Show that if $E \subseteq \mathbb{R}^d$ is Lebesgue measurable, then

$$|E| \;=\; \sup\bigl\{|K| : K \subseteq E,\, K \text{ compact}\bigr\}.$$

The second exercise is to prove a version of the Simple Vitali Lemma for closed balls instead of the version for open balls that we proved in Section 3.4.

Exercise 6 (Simple Vitali Lemma II)**.** Let $\mathcal{B}$ be any collection of closed balls in $\mathbb{R}^d$. Let

$$F \;=\; \bigcup_{B \in \mathcal{B}} B,$$

and fix any $0 < c < |F|$ and any $\varepsilon > 0$. Then there exist disjoint $B_1, \dots, B_k \in \mathcal{B}$ such that

$$\sum_{j=1}^{k} |B_j| \;>\; \frac{c}{(3+\varepsilon)^n}.$$

Hint: Apply the Simple Vitali Lemma for open balls to the collection $\mathcal{C} = \{B^* : B \in \mathcal{B}\}$, where B^* is the ball with the same center as B but with radius $1+\varepsilon$ times larger.

Now we can prove our main result of this part, the *Vitali Covering Lemma*. The proof consists of applying the Simple Vitali Lemma over and over, to each "leftover piece."

Theorem 7 (Vitali Covering Lemma)**.** Suppose that $E \subseteq \mathbb{R}^d$ satisfies $0 < |E|_e < \infty$, and that $\mathcal{B}$ is a Vitali cover of E by closed balls. Then there exist finitely or countably many disjoint balls $\{B_j\}_j$ from $\mathcal{B}$ such that

$$\Big|E \setminus \bigcup_j B_j\Big|_e \;=\; 0 \qquad \text{and} \qquad \sum_j |B_j| \;<\; (1+\varepsilon)\,|E|_e.$$

Proof. Fix

$$\beta \;<\; \min\biggl\{1,\, \frac{|E|_e}{4^n}\biggr\},$$

and choose any $0 < \varepsilon < \beta/2$. Then we can find an open set $U \supseteq E$ such that

$$|U| \;<\; (1+\varepsilon)\,|E|_e.$$

Consider

$$\mathcal{B}_0 \;=\; \{B \in \mathcal{B} : B \subseteq U\}.$$

If $x \in E$ then $x \in U$, so there is some open ball centered at x entirely contained in U, say $B_\rho(x) \subseteq U$. By definition of Vitali cover, there is a closed ball $B \in \mathcal{B}$ that contains x and satisfies radius$(B) < \rho/3$. Then we have $B \subseteq B_\rho(x) \subseteq U$. Therefore $\mathcal{B}_0$ is still a Vitali cover of E, so we can always choose our balls from $\mathcal{B}_0$ from now on.

Now define

$$F \;=\; \bigcup_{B \in \mathcal{B}_0} B.$$

Since $\beta < |E|_e/4^n$, the Simple Vitali Lemma (Exercise 6) implies that there exist disjoint $B_1, \dots, B_{N_1} \in \mathcal{B}_0$ such that

$$\sum_{j=1}^{N_1} |B_j| \;\ge\; \beta.$$

Therefore, since the balls $B_1, \dots, B_{N_1}$ are disjoint and each have finite measure, we have

$$\begin{aligned}
\Big|E \setminus \bigcup_{j=1}^{N_1} B_j\Big|_e \;&\le\; \Big|F \setminus \bigcup_{j=1}^{N_1} B_j\Big| \\
&=\; |F| \;-\; \sum_{j=1}^{N_1} |B_j| \\
&<\; (1+\varepsilon)\,|E|_e \;-\; \beta \\
&<\; \Bigl(1 - \frac{\beta}{2}\Bigr)\,|E|_e.
\end{aligned}$$

Now define

$$E_1 \;=\; E \setminus \bigcup_{j=1}^{N_1} B_j.$$

Note that

$$\sum_{j=1}^{N_1} |B_j| \;=\; \Big|\bigcup_{j=1}^{N_1} B_j\Big| \;\le\; |U| \;<\; (1+\varepsilon)\,|E|_e.$$

Therefore, if $|E_1| = 0$, then we are done. Otherwise, consider

$$\mathcal{B}_1 \;=\; \{B \in \mathcal{B}_0 : B \cap B_j = \emptyset,\, j = 1, \dots, N_1\}.$$

We claim that $\mathcal{B}_1$ is a Vitali cover of E_1. To see this, suppose that $x \in E_1$ and $\eta > 0$ is given. Then x does not belong to the compact set $B_1 \cup \cdots \cup B_{N_1}$, so lies a positive distance δ from this set. Since $\mathcal{B}_0$ is a Vitali cover of E, we can find a ball $B \in \mathcal{B}_0$ that contains x and satisfies

$$\operatorname{radius}(B) \;<\; \min\Big\{\eta, \frac{\delta}{3}\Big\}.$$

Consequently, B is disjoint from each $B_1, \dots, B_{N_1}$, and therefore $B \in \mathcal{B}_1$. Hence $\mathcal{B}_1$ is indeed a Vitali cover of E_1.

As above, we can then find disjoint $B_{N_1+1}, \dots, B_{N_2}$ in $\mathcal{B}_1$ such that

$$\Big|E \setminus \bigcup_{j=1}^{N_2} B_j\Big|_e \;=\; \Big|E_1 \setminus \bigcup_{j=N_1+1}^{N_2} B_j\Big|_e \;<\; \Big(1 - \frac{\beta}{2}\Big)\,|E_1|_e \;<\; \Big(1 - \frac{\beta}{2}\Big)^2\,|E|_e.$$

Again, either

$$E_2 \;=\; E \setminus \bigcup_{j=1}^{N_2} B_j$$

has zero exterior measure and we are done, or we repeat this process again. This procedure either stops after finitely many steps or proceeds forever. In the latter case, at the mth stage we have obtained disjoint sets $B_1, \dots, B_{N_m} \in \mathcal{B}_0$ which satisfy

$$\Big|E \setminus \bigcup_{j=1}^{N_m} B_j\Big|_e \;<\; \Big(1 - \frac{\beta}{2}\Big)^m\,|E|_e.$$

Consequently,

$$\Big|E \setminus \bigcup_{j=1}^{\infty} B_j\Big|_e \;=\; 0.$$

Further, by disjointness we have that

$$\sum_{j=1}^{\infty} |B_j| \;=\; \Big|\bigcup_{j=1}^{\infty} B_j\Big| \;\le\; |U| \;<\; (1+\varepsilon)\,|E|_e. \qquad \square$$

Corollary 8. Suppose that $E \subseteq \mathbb{R}^d$ satisfies $0 < |E|_e < \infty$, and that $\mathcal{B}$ is a Vitali cover of E by closed balls. Then given $\varepsilon > 0$, there exist finitely many disjoint balls $B_1, \dots, B_N \in \mathcal{B}$ such that

$$\Big|E \setminus \bigcup_{j=1}^{N} B_j\Big|_e \;<\; \varepsilon \tag{2}$$

and

$$|E|_e - \varepsilon \;<\; \sum_{j=1}^{N} |B_j| \;<\; (1+\varepsilon)\,|E|_e. \tag{3}$$

Proof. The proof of Theorem 7 shows that we can find disjoint $B_1, \dots, B_N \in \mathcal{B}$ that satisfy both equation (2) and the second inequality in equation (3). To show the first inequality in equation (3), observe that

$$|E|_e \;=\; \Big|E \setminus \bigcup_{j=1}^N B_j\Big|_e \;+\; \Big|E \cap \bigcup_{j=1}^N B_j\Big|_e.$$

Therefore

$$\begin{aligned} \sum_{j=1}^N |B_j| \;&=\; \Big|\bigcup_{j=1}^N B_j\Big| \\ &\geq\; \Big|E \cap \bigcup_{j=1}^N B_j\Big|_e \\ &=\; |E|_e \;-\; \Big|E \setminus \bigcup_{j=1}^N B_j\Big|_e \\ &>\; |E|_e - \varepsilon. \end{aligned}$$

□

3.5.3 Absolutely Continuous Functions on $\mathbb{R}$

Now we turn to absolutely continuous functions on the real line. A collection of intervals in $\mathbb{R}$ are called *nonoverlapping* if their interiors are disjoint.

Definition 9 (Absolutely Continuous Function)**.** We say that a function $f\colon [a,b] \to \mathbb{C}$ is *absolutely continuous on* $[a,b]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite or countably infinite collection of nonoverlapping subintervals $\{[a_j,b_j]\}_j$ of $[a,b]$, we have

$$\sum_j (b_j - a_j) < \delta \quad\Longrightarrow\quad \sum_j |f(b_j) - f(a_j)| < \varepsilon.$$

We define

$$\mathrm{AC}[a,b] \;=\; \big\{f\colon [a,b] \to \mathbb{C} : f \text{ is absolutely continuous on } [a,b]\big\}.$$

The space of *locally absolutely continuous functions on* $\mathbb{R}$ is

$$\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}) \;=\; \big\{f\colon \mathbb{R} \to \mathbb{C} : f \in \mathrm{AC}[a,b] \text{ for every } a < b\big\}.$$

Later, we will see the connection between absolutely continuous functions and absolutely continuous Borel measures.

The next exercise gives some of the basic properties of absolutely continuous functions (recall that $\mathrm{Lip}[a,b]$ denotes the space of functions that are Lipschitz on $[a,b]$, which was defined in earlier sections).

Exercise 10. Prove the following statements.

(a) If $g \in \mathrm{AC}[a,b]$, then g is uniformly continuous on $[a,b]$.

Hint: Consider a single subinterval $\{[x,y]\}$ in the definition of absolutely continuity.

(b) $\mathrm{Lip}[a,b] \subsetneq \mathrm{AC}[a,b] \subsetneq \mathrm{BV}[a,b]$.

Hint: To find an absolutely continuous function that is not Lipschitz, consider Exercise 13. To find a function of bounded variation that is not absolutely continuous, consider the Cantor–Lebesgue function.

Exercise 11. Let $E \subseteq \mathbb{R}$ be measurable, and suppose that $f\colon E \to \mathbb{R}$ is Lipschitz on E, i.e., $|f(x) - f(y)| \le C\,|x-y|$ for all $x, y \in E$. Prove that if $A \subseteq E$, then

$$|f(A)|_e \;\le\; C\,|A|_e. \tag{4}$$

Note that even if A is measurable, it need not be true that $f(A)$ is measurable, which is why we must use exterior Lebesgue measure in (4). Compare this problem to Lemma 18.

An important fact is that absolutely continuity is mutually exclusive with singularity, in the following sense.

Lemma 12. If f is both absolutely continuous and singular on $[a,b]$, then f is constant on $[a,b]$.

Proof. Suppose that f is both absolutely continuous and singular. We will show that $f(a) = f(b)$. Since the same argument can be applied to any subinterval of $[a,b]$, it follows from this that f is constant.

Since f is singular, $E = \{x \in (a,b) : f'(x) = 0\}$ is a set of full measure, i.e., $|E| = b - a$.

Suppose that $x \in E$, and fix any $\varepsilon > 0$. Then since $f'(x) = 0$, we can find an $y_x > 0$ such that we have both $[x, y_x] \subseteq (a,b)$ and

$$x < y < y_x \quad\implies\quad \frac{|f(y) - f(x)|}{y-x} \;<\; \varepsilon.$$

Then

$$\mathcal{B} \;=\; \bigl\{[x,y] : x \in E \text{ and } x < y < y_x\bigr\}$$

is a Vitali cover of E by closed intervals (compare Definition 4).

Let δ be the number corresponding to ε in the definition of absolute continuity (see Definition 9). Applying the Vitali Covering Lemma in the form of Corollary 8, there exist finitely many disjoint intervals $\bigl\{[x_j, y_j]\bigr\}_{j=1}^N$ belonging to $\mathcal{B}$ such that

$$\sum_{j=1}^N (y_j - x_j) \;>\; (b-a) - \delta. \tag{5}$$

Note that the fact that $[x_j, y_j] \in \mathcal{B}$ implies that

$$\frac{|f(y_j) - f(x_j)|}{y_j - x_j} \;<\; \varepsilon, \qquad j = 1, \dots, N. \tag{6}$$

Set $y_0 = a$ and $x_{N+1} = b$. Then we have

$$a \;=\; y_0 \;\le\; x_1 \;<\; y_1 \;<\; x_2 \;<\; \cdots \;<\; y_{N-1} \;<\; x_N \;<\; y_N \;\le\; x_{N+1} \;=\; b.$$

Therefore, considering equation (5), we conclude that

$$\sum_{j=0}^{N} (x_{j+1} - y_j) \ < \ \delta. \tag{7}$$

Since f is absolutely continuous, it follows from equation (7) that that

$$\sum_{j=0}^{N} |f(x_{j+1}) - f(y_j)| \ < \ \varepsilon.$$

On the other hand, equation (6) implies that

$$\sum_{j=1}^{N} |f(y_j) - f(x_j)| \ < \ \varepsilon \sum_{j=1}^{N} (y_j - x_j) \ \leq \ \varepsilon\,(b-a).$$

Hence

$$|f(b) - f(a)| \ \leq \ \sum_{j=0}^{N} |f(x_{j+1}) - f(y_j)| \ + \ \sum_{j=1}^{N} |f(y_j) - f(x_j)| \ \leq \ \varepsilon \ + \ \varepsilon\,(b-a).$$

Since ε is arbitrary, we conclude that $f(a) = f(b)$. □

3.5.4. The Fundamental Theorem of Calculus for Absolutely Continuous Functions

To motivate our next main theorem, we note that the antiderivative of an integrable function is absolutely continuous.

Exercise 13. Show that if $f \in L^1[a,b]$, then $g(x) = \int_a^x f(t)\,dt$ belongs to $\mathrm{AC}[a,b]$, and furthermore $g'(x) = f$ a.e.

Hint: To show absolute continuity, use the earlier exercise that if $f \in L^1(\mathbb{R}^d)$ and $\varepsilon > 0$ are given, then there exists a $\delta > 0$ such that $\int_E |f| < \varepsilon$ for every measurable $E \subseteq \mathbb{R}^d$ with $|E| < \delta$. Then use the Lebesgue Differentiation Theorem to compute g'.

Remark: Note that if g is differentiable everywhere and g' is bounded, then g is a Lipschitz function.

In fact, much more holds.

Theorem 14 (Fundamental Theorem of Calculus for AC Functions)**.** If $g\colon [a,b] \to \mathbb{C}$, then the following statements are equivalent.

(a) $g \in \mathrm{AC}[a,b]$.

(b) There exists $f \in L^1[a,b]$ such that

$$g(x) - g(a) \ = \ \int_a^x f(t)\,dt, \qquad x \in [a,b].$$

(c) g is differentiable almost everywhere, $g' \in L^1[a,b]$, and

$$g(x) - g(a) \;=\; \int_a^x g'(t)\,dt, \qquad x \in [a,b].$$

Proof. (c) $\Rightarrow$ (b). This is immediate.

(b) $\Rightarrow$ (a). This is Exercise 13.

(a) $\Rightarrow$ (c). Suppose that g is absolutely continuous on $[a,b]$. Then g has bounded variation, and so by a result from the first half of the notes on Section 3.5, we know that g' exists a.e. and is integrable. Therefore the function

$$G(x) \;=\; \int_a^x g'$$

is well-defined for each $x \in [a,b]$. Moreover, by the Lebesgue Differentiation Theorem, $G' = g'$ a.e. Hence $(G-g)' = 0$ a.e., so the function $G - g$ is singular on $[a,b]$. On the other hand, both g and G are absolutely continuous on $[a,b]$, so $G - g$ is absolutely continuous as well. Therefore we have by Lemma 12 that $G - g$ is constant (everywhere, since $G - g$ is continuous). Consequently, given any $x \in [a,b]$, we have

$$G(x) - g(x) \;=\; G(a) - g(a) \;=\; 0 - g(a) \;=\; -g(a).$$

Thus $G(x) = g(x) - g(a)$ for all $x \in [0,1]$, so statement (c) holds. □

In particular, if φ is the Cantor–Lebesgue function on $[0,1]$, then φ is singular, and hence is differentiable almost everywhere with $\varphi' \in L^1[a,b]$, yet we have $\varphi(x) - \varphi(0) \neq \int_0^x \varphi'(t)\,dt = 0$, confirming the fact that φ is not absolutely continuous.

We can use Exercise 13 and Lemma 12 to prove the following fundamental decomposition of functions of bounded variation.

Corollary 15. If $f \in \mathrm{BV}[a,b]$, then $f = g + h$ where $g \in \mathrm{AC}[a,b]$ and h is singular on $[a,b]$. Moreover, g and h are unique up to additive constants, and we can take

$$g(x) \;=\; \int_a^x f', \qquad x \in [a,b]. \tag{8}$$

Proof. Since f has bounded variation on $[a,b]$, we know that f' exists a.e. and is integrable. Therefore the function g given by equation (8) is well-defined. Set $h = f - g$. By Exercise 13, we have $g \in \mathrm{AC}[a,b]$ and $g' = f'$ a.e., so $h' = 0$ a.e. Thus h is singular.

If we also had $f = g_1 + h_1$ with g_1 absolutely continuous and h_1 singular, then

$$g - g_1 \;=\; 0 \;=\; h_1 - h,$$

so $g - g_1$ and $h_1 - h$ are each absolutely continuous and singular, and therefore are constant by Lemma 12. □

In the notes on functions of bounded variation, we proved a special case of the following result, requiring that f be differentiable everywhere and f' continuous on $[a,b]$. We can now extend that result to apply to all absolutely continuous functions on $[a,b]$.

Theorem 16. If $f \in \mathrm{AC}[a,b]$, then $V(x) = V[f;a,x]$, $V^+(x) = V^+[f;a,x]$, and $V^-(x) = V^-[f;a,x]$ are each absolutely continuous, and

$$V(x) \;=\; \int_a^x |f'|, \qquad V^+(x) \;=\; \int_a^x (f')^+, \qquad V^-(x) \;=\; \int_a^x (f')^-.$$

Proof. Suppose that $[c,d] \subseteq [a,b]$, and consider the variation of f on $[c,d]$. If $\Gamma = \{c = x_0 < \cdots < x_m = d\}$ is any partition of $[c,d]$, then by Theorem 14, we have

$$S_\Gamma \;=\; \sum_{k=1}^m |f(x_k) - f(x_{k-1})| \;=\; \sum_{k=1}^m \left| \int_{x_{k-1}}^{x_k} f' \right| \;\le\; \sum_{k=1}^m \int_{x_{k-1}}^{x_k} |f'| \;=\; \int_c^d |f'|.$$

Taking the supremum over all such partitions,

$$V(d) - V(c) \;=\; V[f;c,d] \;=\; \sup_\Gamma S_\Gamma \;\le\; \int_c^d |f'|.$$

Suppose now that $\{[a_j,b_j]\}_j$ is any collection of at most countably many nonoverlapping subintervals of $[a,b]$. Then we have

$$\sum_j (V(b_j) - V(a_j)) \;\le\; \int_{\cup[a_j,b_j]} |f'|. \tag{9}$$

Since f has bounded variation, we know that f' is integrable. Therefore, by an earlier exercise, given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f'| < \varepsilon$ whenever $|E| < \delta$. Combining this with equation (9), we see that V is absolutely continuous. Consequently, by the Fundamental Theorem of Calculus for absolutely continuous functions, we have

$$\begin{aligned} V(x) \;&=\; V(x) - V(a) \\ &=\; \int_a^x V' \qquad \text{(Theorem 14)} \\ &=\; \int_a^x |f'| \qquad \text{(since } V' = |f'| \text{ a.e.)} \end{aligned}$$

Exercise: Finish the proof for V^+ and V^-. □

An important fact is that integration by parts is valid for absolutely continuous functions. We will prove a more general version of this result later by making use of Lebesgue–Stieltjes integrals (see Theorem 30 below and also Theorem 7.32 in the text by Wheeden and Zygmund).

Theorem 17 (Integration by Parts)**.** If $f,\, g \in \mathrm{AC}[a,b]$, then

$$\int_a^b f(x)\, g'(x)\, dx \;=\; f(b)g(b) - f(a)g(a) - \int_a^b f'(x)\, g(x)\, dx.$$

3.5.4 The Banach–Zarecki Theorem and Its Relatives

Consider that if $f\colon [a,b] \to \mathbb{C}$ is differentiable everywhere and f' is bounded, then f is Lipschitz by an earlier exercise, and hence f is absolutely continuous on $[a,b]$. Our first goal in this section is to prove the much more subtle fact that if f is differentiable everywhere on $[a,b]$ and we assume only that $f' \in L^1[a,b]$, then f is absolutely continuous. The subtlety here is that the assumptions $f,\ f' \in L^1[a,b]$ do imply that the antiderivative $g(x) = \int_a^x f'(t)\,dt$ exists and is absolutely continuous, but it is not at all obvious that g need equal f.

To prove this, we will need two lemmas. The first lemma is a refinement of Exercise 11. That exercise shows that if a function f is Lipschitz on $[a,b]$ and E is any subset of $[a,b]$, then then $|f(E)|_e \le C\,|E|_e$, where C is a Lipschitz constant for f. In particular, if f is differentiable on $[a,b]$ and f' is bounded on $[a,b]$, then we know that f is Lipschitz, and hence can apply Exercise 11 to this f. However, suppose that instead we know only that f is differentiable and that f' is bounded *on the subset* E. This is not enough to imply that f is Lipschitz on some interval containing E. For example, let f be the Cantor–Lebesgue function on $[0,1]$, and let E be the complement of the Cantor set. Then f is differentiable everywhere on E, and in fact $f'(x) = 0$ for every $x \in E$. However f is not Lipschitz on E or any interval containing E. Hence we cannot apply Exercise 11 in this situation. However, by making the argument a little more sophisticated, the next lemma shows that we still obtain the expected conclusion.

Lemma 18. Let $f\colon [a,b] \to \mathbb{R}$ and $E \subseteq [a,b]$ be given. Suppose that f is differentiable at every point of E, and that

$$M \;=\; \sup_{x\in E} |f'(x)| \;<\; \infty.$$

Then

$$|f(E)|_e \;\le\; M\,|E|_e.$$

Proof. Fix $\varepsilon > 0$. Given $x \in E$, we have

$$\lim_{y\to x} \frac{|f(x)-f(y)|}{|x-y|} \;=\; |f'(x)| \;\le\; M.$$

Hence there exists some $n_x \in \mathbb{N}$ such that if $y \in [a,b]$ then

$$|x-y| < \frac{1}{n_x} \quad\Longrightarrow\quad |f(x)-f(y)| \le (M+\varepsilon)\,|x-y|.$$

Therefore, if for each $n \in \mathbb{N}$ we define

$$E_n \;=\; \Bigl\{x \in E : \text{if } y \in [a,b] \text{ and } |x-y| < \frac{1}{n} \text{ then } |f(x)-f(y)| \le (M+\varepsilon)\,|x-y|\Bigr\},$$

then we have that

$$E \;=\; \bigcup_{n=1}^{\infty} E_n.$$

Further, $E_1 \subseteq E_2 \subseteq \cdots$. Even though the sets E_n need not measurable, we have by an earlier exercise that continuity from below holds for *exterior* Lebesgue measure, so

$$|E|_e \;=\; \lim_{n\to\infty} |E_n|_e.$$

As the sets $f(E_n)$ are also nested increasing and increase to $f(E)$, we also have

$$|f(E)|_e = \lim_{n\to\infty} |f(E_n)|_e.$$

For each n, we can find at most countably many intervals I_n^k such that

$$E \subseteq \bigcup_k I_n^k \qquad \text{and} \qquad \sum_k |I_n^k| \le |E_n|_e + \varepsilon.$$

By subdividing if necessary, we may assume that each interval I_n^k has length less than $\frac{1}{n}$. Therefore, if we take $x, y \in E_n \cap I_n^k$, then we have $|x-y| < \frac{1}{n}$, so

$$|f(x) - f(y)| \le (M+\varepsilon)\,|x-y|.$$

Consequently, $f(E_n \cap I_n^k)$ is contained in an interval of length at most $(M+\varepsilon)\,|I_n^k|$, so

$$|f(E_n \cap I_n^k)|_e \le (M+\varepsilon)\,|I_n^k|.$$

Therefore

$$|f(E_n)|_e \le \sum_k |f(E_n \cap I_n^k)| \le (M+\varepsilon) \sum_k |I_n^k| \le (M+\varepsilon)\,(|E_n|_e + \varepsilon).$$

Hence,

$$|f(E)|_e = \lim_{n\to\infty} |f(E_n)|_e \le (M+\varepsilon) \lim_{n\to\infty} (|E_n|_e + \varepsilon) = (M+\varepsilon)\,(|E|_e + \varepsilon).$$

Since ε is arbitrary, the result follows. □ □

The second lemma relates the measure of $f(E)$ to the integral of $|f'|$ on E. Note that even though we now assume that E is measurable, we cannot conclude that $f(E)$ is measurable, and hence this result must also be formulated in terms of the exterior Lebesgue measure of $f(E)$ (compare Problems 38 and 39, which show that an absolutely continuous function must map measurable sets to measurable sets, but an arbitrary continuous function need not do so).

Lemma 19. Let $f\colon [a,b] \to \mathbb{R}$ be measurable. If $E \subseteq [a,b]$ is measurable and f is differentiable at every point of E, then

$$|f(E)|_e \le \int_E |f'|.$$

Proof. Exercise: Show that the derivative f' is a measurable function (since it is a limit of measurable functions).

For each $k \in \mathbb{N}$, define

$$E_k = \{x \in E : (k-1)\varepsilon \le |f'(x)| < k\varepsilon\}.$$

Since f is differentiable everywhere on E, we have $E = \cup E_k$ disjointly. Further, by Lemma 18 we have that $|f(E_k)|_e \le k\varepsilon\, |E_k|$. Therefore

$$\begin{aligned}
|f(E)|_e \;=\; \left|\bigcup_{k=1}^{\infty} f(E_k)\right|_e \;&\le\; \sum_{k=1}^{\infty} |f(E_k)|_e \\
&\le\; \sum_{k=1}^{\infty} k\varepsilon\, |E_k| \\
&=\; \sum_{k=1}^{\infty} (k-1)\varepsilon\, |E_k| \;+\; \sum_{k=1}^{\infty} \varepsilon\, |E_k| \\
&\le\; \sum_{k=1}^{\infty} \int_{E_k} |f'| \;+\; \varepsilon\, |E| \\
&=\; \int_E |f'| \;+\; \varepsilon\, |E|.
\end{aligned}$$

Since ε is arbitrary, the result follows. □

Theorem 20. If $f \in L^1[a,b] \to \mathbb{C}$ is everywhere differentiable and $f' \in L^1[a,b]$, then $f \in \mathrm{AC}[a,b]$.

Proof. By applying the result to the real and imaginary parts of f, we may assume that f is real-valued.

Choose $\varepsilon > 0$. Since f' is integrable, by an earlier exercise there exists a $\delta > 0$ such that $\int_E |f'| < \varepsilon$ for any measurable set $E \subseteq [a,b]$ with $|E| < \delta$.

Suppose that $\{[a_j, b_j]\}$ is a collection of finitely or countably many nonoverlapping intervals in $[a,b]$ such that $\sum (b_j - a_j) < \delta$. Define $E = \cup\, [a_j, b_j]$, so $|E| < \delta$.

By the Intermediate Value Theorem, $f[a_j, b_j]$ contains the closed interval from $f(a_j)$ to $f(b_j)$, i.e., either $[f(a_j), f(b_j)]$ or $[f(b_j), f(a_j)]$ depending on order. By Lemma 19 we therefore have

$$\sum_j \bigl|f(b_j) - f(a_j)\bigr| \;\le\; \sum_j \bigl|f[a_j, b_j]\bigr| \;\le\; \sum_j \int_{a_j}^{b_j} |f'| \;=\; \int_E |f'| \;<\; \varepsilon. \tag{10}$$

Hence f is absolutely continuous on $[a,b]$. □ □

The hypotheses of Theorem 20 can be relaxed somewhat. For example, if f is differentiable except at *countably* many points and $f' \in L^1[a,b]$, then f will be absolutely continuous. In contrast, the assumptions that f is differentiable a.e. and $f' \in L^1[a,b]$ are not sufficient to ensure that f is absolutely continuous — consider the Cantor–Lebesgue function.

Theorem 20 is closed related to the *Banach–Zarecki Theorem*.

Theorem 21 (Banach–Zarecki Theorem)**.** Let $f\colon [a,b] \to \mathbb{R}$ be given. Then the following statements are equivalent.

(a) $f \in \mathrm{AC}[a,b]$.

(b) f is continuous, $f \in \mathrm{BV}[a,b]$, and $|f(A)| = 0$ for every $A \subseteq [a,b]$ with $|A| = 0$.

Proof. (a) $\Rightarrow$ (b). Suppose that $f \in \mathrm{AC}[a,b]$. Then f is continuous and has bounded variation by Exercise 10, so it remains to show that f maps zero measure sets to zero measure sets.

Suppose that A is a subset of $[a,b]$ of measure zero. Since $\{f(a), f(b)\}$ is a set of measure zero, it suffices to assume that $A \subseteq (a,b)$ with $|A| = 0$. Choose any $\varepsilon > 0$. Since f is absolutely continuous, there exists a $\delta > 0$ such that if $\{[a_j, b_j]\}$ is any collection of nonoverlapping intervals such that $\sum (b_j - a_j) < \delta$, then $\sum |f(b_j) - f(a_j)| < \varepsilon$.

Since A is Lebesgue measurable, we can find an open set $U \supseteq A$ with measure $|U| < |A| + \varepsilon = \varepsilon$, and by intersecting with (a,b) we may assume $U \subseteq (a,b)$. We can write $U = \cup (a_j, b_j)$, a union of at most countably many disjoint open intervals contained in (a,b). Since $[a_j, b_j] \subseteq [a,b]$, there is a point $c_j \in [a_j, b_j]$ where f attains its minimum value on $[a_j, b_j]$, and likewise a point $d_j \in [a_j, b_j]$ where f attains its maximum. Then we have

$$\sum_j |d_j - c_j| \le \sum_j (b_j - a_j) \;<\; \delta,$$

so

$$|f(A)|_e \;\le\; |f(U)|_e \;\le\; \sum_j \bigl|f[a_j, b_j]\bigr|_e \;\le\; \sum_j \bigl|f(d_j) - f(c_j)\bigr| \;<\; \varepsilon.$$

Since ε is arbitrary, we conclude that $|f(A)| = 0$.

(b) $\Rightarrow$ (a). Suppose that statement (b) holds. Since f has bounded variation, it is differentiable a.e. Let D be the set of points where f is differentiable, so $Z = [a,b] \backslash D$ has measure zero.

Suppose that $[c,d] \subseteq [a,b]$, and note that $[f(c), f(d)] \subseteq f[c,d]$. Let $E = [c,d] \cap D$ and $F = [c,d] \backslash D$. Then F has measure zero, so by hypothesis we have $|f(F)| = 0$. Also, f is differentiable everywhere on E, so by Lemma 19 we have

$$|f(d) - f(c)| \;\le\; |f[c,d]|_e \;\le\; |f(E)|_e \,+\, |f(F)|_e \;\le\; \int_E |f'| \,+\, 0 \;=\; \int_c^d |f'|,$$

the final equality following from the fact that E is a subset of $[c,d]$ of full measure.

The remainder of the proof is now similar to the end of the proof of Theorem 20, compare equation (10). □

3.5.5 Functions of Bounded Variation and Complex Borel Measures

Now we explore the connection between functions f that are absolutely continuous, and absolute continuity of the corresponding Borel measure μ_f with respect to Lebesgue measure.

Definition 22. The space of *functions of normalized bounded variation* is

$$\mathrm{NBV}(\mathbb{R}) \;=\; \bigl\{f \in \mathrm{BV}(\mathbb{R}) : f \text{ is right-continuous and } f(-\infty) = 0\bigr\}.$$

Note that NBV($\mathbb{R}$) is a complex vector space, i.e., it is closed under function addition and scalar multiplication. By making use of our earlier results about functions of bounded variation, we see in the next exercise how to "convert" an arbitrary function of bounded variation into a function of normalized bounded variation.

Exercise 23. Let $f \in \mathrm{BV}(\mathbb{R})$ be given, and set $g(x) = f(x+)$.

(a) Show that $g \in \mathrm{BV}(\mathbb{R})$ and $g' = f'$ a.e.

Hint: Break into real and imaginary parts, and then use the fact that every real-valued function of bounded variation can be written as a difference of two monotone increasing functions.

(b) Show that $h(x) = f(x+) - f(-\infty) \in \mathrm{NBV}(\mathbb{R})$.

The following exercise, giving examples of functions in NBV($\mathbb{R}$), is essentially a rewording of things that we did when we constructed Lebesgue–Stieltjes measures in Section 1.5.

Exercise 24. Suppose that μ is a bounded positive Borel measure on $\mathbb{R}$. Show that $f(x) = \mu(-\infty, x] \in \mathrm{NBV}(\mathbb{R})$.

To recall a little more, we showed in Section 1.5 that if $f\colon \mathbb{R} \to \mathbb{R}$ is any nonnegative, *monotone increasing*, right-continuous function, then there exists a unique positive Borel measure μ_f that has the property that

$$\mu_f(a,b] \;=\; f(b) - f(a), \qquad -\infty < a < b < \infty.$$

The positive measure μ_f is the *Lebesgue–Stieltjes measure* associated with f. We will see in the next exercise that *arbitrary* functions $f \in \mathrm{NBV}(\mathbb{R})$ (which need not be monotone increasing) are associated with unique *complex* Borel measures ν_f on $\mathbb{R}$.

Exercise 25. (a) Show that if ν is a complex Borel measure on $\mathbb{R}$ and $f(x) = \nu(-\infty, x]$, then $f \in \mathrm{NBV}(\mathbb{R})$.

Hint: The case $\nu \ge 0$ is Exercise 24. Extend to arbitrary complex measures by writing $\nu = (\nu_r^+ - \nu_r^-) + i(\nu_i^+ - \nu_i^-)$ where $\nu_r^\pm \ge 0$ and $\nu_i^\pm \ge 0$.

(b) Show that if $f \in \mathrm{NBV}(\mathbb{R})$, then there exists a unique complex Borel measure ν_f on $\mathbb{R}$ such that

$$f(x) \;=\; \nu_f(-\infty, a].$$

Further, this measure is regular.

Hint: Again, break into cases by writing $f = (f_r^+ - f_r^-) + i(f_i^+ - f_i^-)$ where each of $f_r^\pm$, $f_i^\pm$ are monotone increasing. Since each of the associated Lebesgue–Stieltjes measures are regular, it follows that ν_f is regular as well.

Given $f \in \mathrm{NBV}(\mathbb{R})$ and its associated Borel measure ν_f, we will characterize the total variation measure $|\nu_f|$ in terms of the variation of f on $(-\infty, x]$. For simplicity of notation, we set

$$V_f(x) \;=\; V[f; -\infty, x] \;=\; \sup_{a<x} V[f; a, x].$$

Lemma 26. If $f \in \mathrm{BV}(\mathbb{R})$ is right-continuous (for example, if $f \in \mathrm{NBV}(\mathbb{R})$), then V_f is right-continuous as well.

Proof. Choose any $x \in \mathbb{R}$ and $\varepsilon > 0$, and define

$$\alpha \;=\; V_f(x+) - V_f(x).$$

Our goal is to show that $\alpha = 0$.

Since V_f is monotone increasing, the function $W(x) = V_f(x+)$ is right-continuous. Since f is also right-continuous, we can find a $\delta > 0$ such that

$$x < y \le x + \delta \qquad \Longrightarrow \qquad |f(y) - f(x)| < \varepsilon \quad \text{and} \quad |V_f(y) - V_f(x+)| < \varepsilon.$$

In particular, we have

$$V_f(x+\delta) - V_f(x) \;=\; \bigl(V_f(x+) - V_f(x)\bigr) + \bigl(V_f(x+\delta) - V_f(x+)\bigr) \;\le\; \alpha + \varepsilon. \tag{11}$$

Now, by definition of total variation, there exists a partition

$$\Gamma_1 \;=\; \{x = x_0 < x_1 < \cdots < x_n = x + \delta\}$$

for which we have $S_{\Gamma_1} \ge \frac{3}{4}\, V[f; x, x+\delta]$. Therefore

$$\begin{aligned}
\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \;=\; S_{\Gamma_1} \;&\ge\; \frac{3}{4}\, V[f; x, x+\delta] \\
&=\; \frac{3}{4}\bigl(V_f(x+\delta) - V_f(x)\bigr) \\
&\ge\; \frac{3}{4}\bigl(V_f(x+) - V_f(x)\bigr) \qquad \text{since } V_f \text{ is increasing} \\
&=\; \frac{3}{4}\,\alpha.
\end{aligned}$$

Since $x = x_0 < x_1 \le x + \delta$, we have $|f(x_1) - f(x_0)| < \varepsilon$, and therefore

$$\sum_{j=2}^{n} |f(x_j) - f(x_{j-1})| \;\ge\; \frac{3}{4}\,\alpha - \varepsilon.$$

Applying then the same reasoning as above to the interval $[x, x_1]$ instead of $[x, x+\delta]$, we can find a partition

$$\Gamma_2 \;=\; \{x = t_0 < x_1 < \cdots < t_m = x_1\}$$

for which

$$\sum_{j=1}^{m} |f(t_j) - f(t_{j-1})| \;\ge\; \frac{3}{4}\,\alpha.$$

Then

$$\Gamma \;=\; \Gamma_1 \cup \Gamma_2 \;=\; \{x = t_0 < x_1 < \cdots < t_m = x_1 < x_2 < \cdots < x_m = x + \delta\}$$

is a partition of $[x, x+\delta]$, so we have

$$\begin{aligned}
\alpha + \varepsilon \;&>\; V_f(x+\delta) - V_f(x) \qquad \text{by equation (11)} \\
&=\; V[f; x, x+\delta] \\
&\geq\; S_\Gamma \\
&=\; \sum_{j=1}^{m} |f(t_j) - f(t_{j-1})| + \sum_{j=2}^{n} |f(x_j) - f(x_{j-1})| \\
&\geq\; \frac{3}{4}\alpha + \frac{3}{4}\alpha - \varepsilon \;=\; \frac{3}{2}\alpha - \varepsilon.
\end{aligned}$$

Rearranging yields $0 \leq \alpha < 4\varepsilon$, so since ε is arbitrary we conclude that $\alpha = 0$. □

Theorem 27. If $f \in \mathrm{NBV}(\mathbb{R})$ and ν_f is its associated complex Borel measure ν_f, then its total variation measure $|\nu_f|$ is the positive Lebesgue–Stieltjes measure associated with the total variation function V_f, i.e.,

$$|\nu_f| \;=\; \nu_{V_f}. \tag{12}$$

Consequently, ν_f is a bounded measure.

Proof. Define

$$g(x) \;=\; |\nu_f|(-\infty, x].$$

Note that g is monotone increasing and right-continuous, and its associated Lebesgue–Stieltjes measure is $\nu_g = |\nu_f|$. Thus, if we show that $g = V_f$, then equation (12) will follow.

Choose any $x \in \mathbb{R}$, and fix any $a < x$. Let $\Gamma = \{a = x_0 < \cdots < x_m = x\}$ be any partition of $[a, x]$. Then we have

$$\begin{aligned}
S_\Gamma \;=\; \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \;&=\; \sum_{j=1}^{n} |\nu_f(x_{j-1}, x_j]| \\
&\leq\; \sum_{j=1}^{n} |\nu_f|(x_{j-1}, x_j] \\
&=\; |\nu_f|(a, x] \\
&\leq\; |\nu_f|(-\infty, x] \;=\; g(x).
\end{aligned}$$

Taking the supremum over all such partitions Γ, we see that $V[f; a, x] \leq g(x)$. Taking then the supremum over all $a < x$ gives $V_f(x) \leq g(x)$.

To obtain the opposite inequality, note that if we choose any $a < b$, then

$$|\nu_f(a, b]| \;=\; |f(b) - f(a)| \;\leq\; V[f; a, b] \;=\; V_f(b) - V_f(a) \;=\; \nu_{V_f}(a, b].$$

Exercise: Extend this inequality to all Borel sets, i.e., show that $|\nu_f(E)| \le \nu_{V_f}(E)$ for each Borel set E. Then use one of the equivalent characterizations of the total variation measure to show that $|\nu_f|(E) \le \nu_{V_f}(E)$ for all Borel sets. Finally, use this to show that $g(x) \le V_f(x)$.

Consequently, we have that $|\nu_f| = \nu_g = \nu_{V_f}$. Since

$$\nu_{V_f}(\mathbb{R}) \;=\; \lim_{x\to\infty} V_f(x) \;=\; V[f;\mathbb{R}] \;<\; \infty,$$

we conclude that ν_f is a bounded measure. □

Our next goal is to characterize the functions $f \in \mathrm{NBV}(\mathbb{R})$ for which the associated measure ν_f is either absolutely continuous or singular with respect to Lebesgue measure dx.

Theorem 28. Let $f \in \mathrm{NBV}(\mathbb{R})$ be given, and let ν_f be the associated complex Borel measure constructed in Exercise 25(b).

(a) $f' \in L^1(\mathbb{R})$, and $\nu_f = f'\,dx + \lambda$ where $\lambda \perp dx$.

(b) $\nu_f \perp dx \iff f' = 0$ a.e.

(c) The following are equivalent:

i. $\nu_f \ll dx$,

ii. $\nu_f = f'\,dx$,

iii. $f(x) = \int_{-\infty}^{x} f'(t)\,dt,\ x \in \mathbb{R}$,

iv. f is absolutely continuous.

Proof. (a) We know that f is differentiable almost everywhere. Further, the associated complex Borel measure ν_f is regular. Let

$$\nu_f \;=\; g\,dx + \lambda$$

be the Lebesgue–Radon–Nikodym decomposition of ν_f with respect to Lebesgue measure, and note that $g \in L^1(\mathbb{R})$ since ν_f is a bounded measure. By a theorem from Section 3.4, we have that

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \;=\; \lim_{h\to 0^+} \frac{\nu_f(x,x+h]}{|(x,x+h]|} \;=\; g(x) \quad \text{a.e.}$$

Applying a similar argument as $h \to 0^-$, we see that $f' = g$ a.e. Therefore $f \in L^1(\mathbb{R})$, and we have

$$\nu_f \;=\; f'\,dx + \lambda. \tag{13}$$

(b) Considering equation (13) and the fact that the Lebesgue–Radon–Nikodym decomposition is unique, we have

$$\begin{aligned} \nu_f \perp dx &\iff \nu_f = \lambda \\ &\iff f'\,dx = 0 \quad \text{(the zero measure on } \mathbb{R}) \\ &\iff f' = 0 \text{ a.e.} \end{aligned}$$

(c) By part (a), the Lebesgue–Radon–Nikodym decomposition of ν_f with respect to Lebesgue measure is given by equation (13).

i $\Rightarrow$ ii. If $\nu_f \ll dx$, then again applying the uniqueness of the Since Lebesgue–Radon–Nikodym decomposition, we have $\lambda = 0$ and $\nu_f = f'\,dx$.

ii $\Rightarrow$ iii. If $\nu_f = f'\,dx$, then, since f' is integrable by part (a),

$$\int_{-\infty}^{x} f' \;=\; \nu_f(-\infty, x], \quad x \in \mathbb{R}$$

By construction we also have $\nu_f(-\infty, x] = f(x)$, see Exercise 25. Therefore we have $f(x) = \int_{-\infty}^{x} f'$ for every x.

The remaining implications are exercises, following from the fact that $f \in \mathrm{NBV}(\mathbb{R})$ and the Fundamental Theorem of Calculus for absolutely continuous functions. □

3.5.6 Lebesgue–Stieltjes Integrals

Definition 29. If $f \in \mathrm{NBV}(\mathbb{R})$, then (if it exists) the integral of a function $g\colon \mathbb{R} \to \mathbb{C}$ with respect to the measure ν_f is called the *Lebesgue–Stieltjes integral of g with respect to f*. It is usually denoted by the shorthands

$$\int g\,df \;=\; \int g(x)\,df(x) \;=\; \int g(x)\,d\nu_f(x).$$

For example, if g is bounded, then since ν_f is a bounded measure we know that $\int g\,df$ will exist.

We can prove an integration by parts formula for Lebesgue–Stieltjes integrals.

Theorem 30. If f, $g \in \mathrm{NBV}(\mathbb{R})$ and at least one of f or g are continuous, then for any $-\infty < a < b < \infty$,

$$\int_{(a,b]} f\,dg \;=\; f(b)g(b) - f(a)g(a) - \int_{(a,b]} g\,df. \tag{14}$$

Proof. Since the formulation of the result is symmetric in f and g, we can assume that g is continuous. Let

$$E \;=\; \{(x, y) \in \mathbb{R}^2 : a < x \le y \le b\}.$$

This is a Borel subset of $\mathbb{R}^2$. Further, since ν_f and ν_g are bounded measures, $\chi_E \in L^1(\nu_f \times \nu_g)$. Therefore, by Fubini's Theorem, we can write $(\nu_f \times \nu_g)(E)$ in two ways.

First, we have

$$\begin{aligned}
(\nu_f \times \nu_g)(E) \;&=\; \int_{(a,b]\times(a,b]} \chi_E\,d(\nu_f \times \nu_g) \\
&=\; \int_{(a,b]} \int_{(a,y]} d\nu_f(x)\,d\nu_g(y) \\
&=\; \int_{(a,b]} \nu_f(a, y]\,d\nu_g(y)
\end{aligned}$$

$$= \int_{(a,b]} \big(f(y) - f(a)\big)\, d\nu_g(y)$$

$$= \int_{(a,b]} f(y)\, d\nu_g(y) \;-\; f(a)\,\big(g(b) - g(a)\big). \tag{15}$$

Exercise: Show that the fact that g is continuous implies that $\nu_g[x,b] = g(b) - g(x)$. Therefore, we have a second way to write $(\nu_f \times \nu_g)(E)$, namely,

$$\begin{aligned}
(\nu_f \times \nu_g)(E) \;&=\; \int_{(a,b]} \int_{[x,b]} d\nu_g(y)\, d\nu_f(x) \\
&=\; \int_{(a,b]} \nu_g[x,b]\, d\nu_f(x) \\
&=\; \int_{(a,b]} \big(g(b) - g(x)\big)\, d\nu_f(x) \\
&=\; g(b)\,\big(f(b) - f(x)\big) - \int_{(a,b]} g(x)\, d\nu_f(x).
\end{aligned} \tag{16}$$

Since equation (15) must equal equation (16), after rearranging, we find that equation (14) holds. □

As a consequence, we obtain the integration by parts formula for absolutely continuous functions stated earlier in Theorem 17.

Exercise 31 (Integration by Parts)**.** If $f,\, g \in \mathrm{AC}[a,b]$, then

$$\int_a^b f(x)\, g'(x)\, dx \;=\; f(b)g(b) - f(a)g(a) - \int_a^b f'(x)\, g(x)\, dx.$$

Hint: Use the fact that $\nu_f = f'\, dx$ and $\nu_g = g'\, dx$.

Remark 32. Just as the Lebesgue integral generalizes the Riemann integral, the Lebesgue–Stieltjes integral generalizes something called the *Riemann–Stieltjes integral*. A direct way of defining the Riemann-Stieltjes integral is as follows.

Let $f,\, g\colon [a,b] \to \mathbb{R}$ be given. Given a partition

$$\Gamma \;=\; \{a = x_0 < x_1 < \cdots < x_m = b\}$$

of $[a,b]$, and given any choice of points $\xi_j \in [x_{j-1}, x_j]$, set

$$R_\Gamma^\xi \;=\; \sum_{j=1}^m g(\xi_j)\,\big(f(x_j) - f(x_{j-1})\big).$$

If

$$\lim_{|\Gamma| \to 0} R_\Gamma^\xi \;=\; I \tag{17}$$

exists, then I is the Riemann–Stieltjes integral of g with respect to f on $[a,b]$, denoted

$$I = \int_a^b g(x)\,df(x) = \int_a^b g\,df,$$

where we use the same notation as for Lebesgue–Stieltjes integrals.

To be more precise, the meaning of the limit in equation (17) is that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition Γ with mesh size $|\Gamma| < \delta$ and for any choice of points $\xi_j \in [x_{j-1}, x_j]$ we have

$$|I - R_\Gamma^\xi| < \varepsilon.$$

Exercise 33. Show that if g is continuous on $[a,b]$ and f is continuously differentiable (differentiable everywhere with a continuous derivative f') then (as a Riemann–Stieltjes integral)

$$\int_a^b g\,df = \int_a^b g(x)\,f'(x)\,dx.$$

Note that this coincides with the Lebesgue–Stieltjes integral of g with respect to f.

For more on Riemann–Stieltjes integrals, see the text by Wheeden and Zygmund.

Additional Problems

Problem 34. Show that the Cantor–Lebesgue function is continuous and has bounded variation on $[0,1]$, but is not absolutely continuous on $[0,1]$.

Problem 35. Define $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $g(0) = 0$. Show that $g \in L^1[a,b]$ is everywhere differentiable, $g' \notin L^1[a,b]$, and $g \notin \mathrm{AC}[a,b]$.

Problem 36. Suppose that $f\colon [a,b] \to \mathbb{C}$ is continuous and differentiable a.e., and $f' \in L^1[a,b]$. Show that if $|f(A)| = 0$ for every $A \subseteq [a,b]$ with $|A| = 0$, then f is absolutely continuous.

Problem 37. Give an example that shows that the hypothesis in Problem 36 that f maps measure zero sets to measure zero sets is necessary.

Problem 38. Show that if $f\colon [a,b] \to \mathbb{C}$ is absolutely continuous and $E \subseteq [a,b]$ is measurable, then $|f(E)|$ is measurable as well.

Problem 39. Show that a continuous function need not map a measurable set to a measurable set.