Hints for Exercises and Additional Problems

Additional Problems from Chapter 2

2.5 Let $\mathcal{U}$ denote the set of all open subsets of $\mathbb{R}$. Any $U \in \mathcal{U}$ can be written as $U = \bigcup (a_k, b_k) \in \Sigma(\mathcal{E}_1)$, so $\mathcal{U} \subseteq \Sigma(\mathcal{E}_1)$. But $\Sigma(\mathcal{U})$ is the smallest $\sigma$-algebra that contains $\mathcal{U}$, so $\mathcal{B}_R = \Sigma(\mathcal{U}) \subseteq \Sigma(\mathcal{E}_1)$.

(b) Given $a < b$,

$$[a,b] = \bigcap_{k=1}^{\infty} (a - \frac{1}{k}, b + \frac{1}{k}) \in \Sigma(\mathcal{E}_1).$$

Hence $\mathcal{E}_2 \subseteq \Sigma(\mathcal{E}_1)$, so $\Sigma(\mathcal{E}_2) \subseteq \Sigma(\mathcal{E}_1)$.

(f) Given $a \in \mathbb{R}$,

$$[a, \infty) = \bigcap_{r \in \mathbb{Q}, r < a} (r, \infty) \in \Sigma(\mathcal{E}_6).$$

2.7 If $\Sigma$ contains infinitely many disjoint sets $E_1, E_2, \ldots$ then $\Sigma$ must be uncountable. One method of showing the existence of such sets is to define a relation on $X$ by declaring $x \sim y$ if and only if

$$\forall A \in \Sigma, \ x \in A \iff y \in A.$$ 

Prove that $\sim$ is an equivalence relation, and show that if $\Sigma$ is countable then the equivalence classes $[x] = \cap \{ A \in \Sigma : x \in A \}$ all belong to $\Sigma$.

2.9 Let $X = \{x_n\}_{n \in \mathbb{N}}$. Let $\Sigma_N$ consist of every set $A \in \mathcal{P}(\{x_1, \ldots, x_N\})$ together with the complement of each such set $A$.

2.17 Write $E = \bigcup E_n$ where $E_n = \{x \in X : \mu\{x\} > \frac{1}{n}\}$. How many points can be in $E_n$?

2.19 Define

$$C = \sup\{\mu(F) : F \in \Sigma, F \subseteq E, \mu(F) < \infty\}.$$ 

If $C < \infty$ then there exist measurable sets $F_k \subseteq E$ with finite measure such that $\mu(F_k) \to C$. Consider the sets $F = \bigcup F_k$ and $A = E \setminus F$. 