**Lusin’s Theorem**

**Theorem 8** (Lusin’s Theorem). Given a measurable set $E \subseteq \mathbb{R}^d$ and given $f : E \to \mathbb{C}$, the following statements are equivalent.

(a) $f$ is measurable.

(b) For each $\varepsilon > 0$, there exists a closed set $F \subseteq E$ with $|E \setminus F| < \varepsilon$ such that $f|_F$ is continuous, i.e.,

$$\forall x_k, x \in F, \ x_k \to x \implies f(x_k) \to f(x).$$

**Proof.** (a) $\Rightarrow$ (b). First we prove the result for simple functions. Suppose that $\phi = \sum_{j=1}^N a_j \chi_{E_j}$ is a simple function, and that the $E_j$ are disjoint. Fix $\varepsilon > 0$. Since $E_j$ is measurable, there exists a closed $F_j \subseteq E_j$ such that

$$|E_j \setminus F_j| < \frac{\varepsilon}{n}, \quad j = 1, \ldots, n.$$

Then

$$F = \bigcup_{j=1}^n E_j$$

is closed, and $|E \setminus F| < \varepsilon$.

If $E$ is a bounded set, then the $F_j$ are compact, and hence

$$\text{dist}(F_j, F_k) > 0$$

if $j \neq k$. Since $\phi$ is constant on each $F_j$, it follows that $\phi|_F$ is continuous.

Exercise: Extend to the case where $E$ is not bounded by considering the sets

$$E_k = \{ x \in E : \| x \| \leq k \}.$$

Now let $f : E \to \mathbb{C}$ be an arbitrary measurable function. Let $\phi_n$ be simple functions such that $\phi_n(x) \to f(x)$ for each $x \in E$. Fix $\varepsilon > 0$. By the previous case, for each $n$ we can find a closed $F_n \subseteq E$ such that

$$|E \setminus F_n| < \frac{\varepsilon}{2^{n+1}}$$

and $\phi_n|_{F_n}$ is continuous.

Suppose that $E$ is bounded. Then by Egoroff’s Theorem, there exists a closed $F_0 \subseteq E$ such that

$$|E \setminus F_0| < \frac{\varepsilon}{2}$$

and $f_n$ converges to $f$ uniformly on $F_0$. Define

$$F = \bigcap_{n=0}^\infty F_n.$$

Then $F$ is closed since each $F_n$ is closed, and

$$|E \setminus F| = \left| \bigcup_{n=0}^\infty (E \setminus F_n) \right| \leq \sum_{n=0}^\infty |E \setminus F_n| \leq \varepsilon.$$
Since \( \phi_n|_F \) is continuous, \( \phi_n|_F \) is continuous as well. And since \( \phi_n \) converges to \( f \) uniformly on \( F \), we have that \( f|_F \) is continuous. This completes the proof for the case that \( E \) is bounded.

Exercise: Extend to the case where \( E \) is unbounded by considering the sets
\[
E_k = \{ x \in E : k - 1 \leq \| x \| < k \}.
\]

(b) \( \Rightarrow \) (a). Suppose that statement (b) holds. By considering the real and imaginary parts of \( f \) separately, it suffices to assume that \( f \) is real-valued.

By hypothesis, for each \( n \in \mathbb{N} \) there exists a closed \( F_n \subseteq E \) such that
\[
|E \setminus F_n| < \frac{1}{n}
\]
and \( f|_{F_n} \) is continuous. Set
\[
H = \bigcup_{n=1}^{\infty} F_n.
\]
Then \( H \) is an \( F_\sigma \)-set, so is measurable. Also, for every \( n \) we have that
\[
|E \setminus H| \leq |E \setminus F_n| < \frac{1}{n},
\]
so \( |E \setminus H| = 0 \). Therefore we can write \( E = H \cup Z \) where \( Z \) has measure zero and is disjoint from \( H \).

If we fix any \( a \in \mathbb{R} \), then we have that
\[
\{ f > a \} = \{ x \in H : f(x) > a \} \cup \{ x \in Z : f(x) > a \}
\]
\[
= \bigcup_{n=1}^{\infty} \{ x \in F_n : f(x) > a \} \cup \{ x \in Z : f(x) > a \}.
\]
Since each \( f|_{F_n} \) is continuous, we have that \( \{ x \in F_n : f(x) > a \} \) is relatively open with respect to \( F_n \) (i.e., it is the intersection of an open set \( U \subseteq \mathbb{R}^d \) with \( F_n \)) and hence is measurable. And since Lebesgue measure is complete, we know that \( \{ x \in Z : f(x) > a \} \) is measurable. Therefore we conclude that \( \{ f > a \} \) is measurable. Since this is true for every real number \( a \), we have shown that \( f \) is a measurable function. \( \square \)