Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Let $X$ be a normed space. Prove that $X$ is a Banach space if and only if every absolutely convergent series in $X$ converges in $X$.

**Definition 1.** We say that a function $f: \mathbb{R} \to \mathbb{C}$ is H"older continuous with exponent $\alpha > 0$ if there exists a constant $K > 0$ such that

$$\forall x, y \in \mathbb{R}, \quad |f(x) - f(y)| \leq K|x - y|^\alpha.$$ 

2. Given $0 < \alpha < 1$, define

$$C^\alpha(\mathbb{R}) = \{ f \in C(\mathbb{R}) : f \text{ is Hölder continuous with exponent } \alpha \}.$$ 

Show that $\|f\|_{C^\alpha} = |f(0)| + \sup_{x \neq y} \frac{f(x) - f(y)}{|x - y|^\alpha}$ is a norm on $C^\alpha(\mathbb{R})$, and that $C^\alpha(\mathbb{R})$ is complete with respect to this norm.

3. Show that if $|E| < \infty$ and $0 < p \leq q \leq \infty$, then $L^q(E) \subseteq L^p(E)$, with

$$\|f\|_p \leq \|f\|_q |E|^{\frac{1}{p} - \frac{1}{q}}, \quad f \in L^p(\mathbb{R}).$$

In contrast, show that $L^p(\mathbb{R})$ is not contained in $L^q(\mathbb{R})$ for any $p \neq q$, regardless of which is larger.

4. Suppose that $E \subseteq \mathbb{R}^d$ is Lebesgue measurable and satisfies $|E| < \infty$. Show that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

5. Show that $\ell^p$ is separable when $p < \infty$. 