

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Let X, Y be Banach spaces, and let $A \in \mathcal{B}(X, Y)$ be fixed. Show that there exists a unique operator $A^* \in \mathcal{B}(Y^*, X^*)$ that satisfies

$$\forall x \in X, \quad \forall \mu \in Y^*, \quad \langle Ax, \mu \rangle = \langle x, A^* \mu \rangle. \quad (1)$$

Show further that

$$\|A^*\| = \|A\|.$$

2. Show that the Baire Category Theorem is equivalent to the following statement: If X is a complete metric space and $U_n \subseteq X$ is dense and open for $n \in \mathbb{N}$, then $\bigcap U_n$ is dense in X .

3. Show that if X is an infinite-dimensional Banach space, then any Hamel basis for X must be uncountable.

Remark: A Hamel basis is an ordinary vector space basis, i.e., its finite linear span is X and every finite subset is linearly independent.

Definition. Let X be a Banach space, and let f_n, f be vectors in X . Then we say that f_n converges weakly to f , denoted $f_n \xrightarrow{w} f$, if

$$\forall \mu \in X^*, \quad \lim_{n \rightarrow \infty} \langle f_n, \mu \rangle = \langle f, \mu \rangle.$$

Definition/Theorem. $M_b(\mathbb{R})$ is the space of all complex Borel measures on \mathbb{R} . This is a Banach space with respect to the norm $\|\nu\| = |\nu|(\mathbb{R})$, where $|\nu|$ is the total variation measure of ν .

Riesz Representation Theorem $C_0(\mathbb{R})^* \cong M_b(\mathbb{R})$. Specifically each complex measure $\nu \in M_b(\mathbb{R})$ defines a bounded linear functional on $C_0(\mathbb{R})$ via

$$\langle f, \nu \rangle = \int f(x) d\bar{\nu}(x) = \int \overline{f(x)} d\nu(x), \quad f \in C_0(\mathbb{R}),$$

and every bounded linear functional on $C_0(\mathbb{R})$ has this form for some measure $\nu \in M_b(\mathbb{R})$.

4. Let $f_n, f \in C_0(\mathbb{R})$. Show that $f_n \xrightarrow{w} f$ (weak convergence) in $C_0(\mathbb{R})$ if and only if $f_n(x) \rightarrow f(x)$ pointwise for each x and $\sup \|f_n\|_\infty < \infty$.