1. Let $X, Y$ be Banach spaces, and let $A \in B(X, Y)$ be fixed. Show that there exists a unique operator $A^* \in B(Y^*, X^*)$ that satisfies
\[\forall x \in X, \quad \forall \mu \in Y^*, \quad \langle Ax, \mu \rangle = \langle x, A^*\mu \rangle.\]  
(1)

Show further that $\|A^*\| = \|A\|$.

2. Show that the Baire Category Theorem is equivalent to the following statement: If $X$ is a complete metric space and $U_n \subseteq X$ is dense and open for $n \in \mathbb{N}$, then $\cap U_n$ is dense in $X$.

3. Show that if $X$ is an infinite-dimensional Banach space, then any Hamel basis for $X$ must be uncountable.

Remark: A Hamel basis is an ordinary vector space basis, i.e., its finite linear span is $X$ and every finite subset is linearly independent.

**Definition.** Let $X$ be a Banach space, and let $f_n, f$ be vectors in $X$. Then we say that $f_n$ converges weakly to $f$, denoted $f_n \overset{w}{\to} f$, if
\[\forall \mu \in X^*, \quad \lim_{n \to \infty} \langle f_n, \mu \rangle = \langle f, \mu \rangle.\]

**Definition/Theorem.** $M_b(\mathbb{R})$ is the space of all complex Borel measures on $\mathbb{R}$. This is a Banach space with respect to the norm $\|\nu\| = |\nu|(\mathbb{R})$, where $|\nu|$ is the total variation measure of $\nu$.

**Riesz Representation Theorem.** $C_0(\mathbb{R})^* \cong M_b(\mathbb{R})$. Specifically each complex measure $\nu \in M_b(\mathbb{R})$ defines a bounded linear functional on $C_0(\mathbb{R})$ via
\[\langle f, \nu \rangle = \int f(x) d\nu(x) = \int \overline{f(x)} d\nu(x), \quad f \in C_0(\mathbb{R}),\]
and every bounded linear functional on $C_0(\mathbb{R})$ has this form for some measure $\nu \in M_b(\mathbb{R})$.

4. Let $f_n, f \in C_0(\mathbb{R})$. Show that $f_n \overset{w}{\to} f$ (weak convergence) in $C_0(\mathbb{R})$ if and only if $f_n(x) \to f(x)$ pointwise for each $x$ and $\sup \|f_n\|_\infty < \infty$. 

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