MATH 6338 HOMEWORK #5 Due date: July 23, 2008

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

Definition. A set of vectors $\{f_n\}_{n\in\mathbb{N}}$ in a Hilbert space H is a *tight frame* for H if there exists a number $A > 0$ so that

$$\forall\, f \in H, \quad \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \;=\; A\,\|f\|^2.$$

The number A is the *frame bound*. The *frame operator* is the mapping $S : H \to H$ defined by

$$Sf \;=\; \sum_{n=1}^{\infty} \langle f, f_n\rangle\, f_n \quad \text{for } f \in H.$$

We proved in class that this series converges for each f (this is a consequence of the fact that $\sum |\langle f, f_n\rangle|^2 < \infty$). □

1. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a tight frame with frame bound A.

(a) Show directly that S and $S - AI$ are self-adjoint, where I is the identity operator.

(b) Show that $S = AI$.
Hint: The norm of a self-adjoint operator T on a Hilbert space H can be computed using the formula

$$\|T\| \;=\; \sup_{\|f\|=1} |\langle Tf, f\rangle|.$$

(c) Show directly that the following three statements are equivalent.

i. $\|f_n\|^2 = A$ for every n.

ii. $\{f_n\}_{n\in\mathbb{N}}$ is an orthogonal (but not necessarily orthonormal) sequence with no zero elements. That is, $\langle f_m, f_n\rangle = 0$ if $m \neq n$, and every $f_n \neq 0$.

iii. $\{f_n\}_{n\in\mathbb{N}}$ is a Schauder basis for H.

2. (a) Show that if $T > 1$ then $\{e^{2\pi inTx}\}_{n\in\mathbb{Z}}$ is incomplete in $L^2[0,1]$.

(b) Show that if $0 < T < 1$, then $\{e^{2\pi inTx}\}_{n\in\mathbb{Z}}$ is a tight frame for $L^2[0,1]$. What is the frame bound?

(c) Show that $\{e^{2\pi inTx}\}_{n\in\mathbb{Z}}$ is not a Schauder basis for $L^2[0,1]$ when $0 < T < 1$. In particular, find two different ways to write the constant function 1 (on the domain $[0,1]$) as an infinite linear combination of the exponentials $e^{2\pi inTx}$.

3. (a) Prove the following *perturbation result* for frames. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a frame for a Hilbert space H with frame bounds A, B. This means that $\{f_n\}_{n\in\mathbb{N}}$ is not necessarily tight, instead, we have that

$$\forall\, f \in H, \quad A\,\|f\|^2 \;\le\; \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \;\le\; B\,\|f\|^2.$$

Suppose also that $\{g_n\}$ is such that $\{f_n - g_n\}_{n\in\mathbb{N}}$ satisfies an *upper* frame bound condition of the form

$$\forall\, f \in H, \quad \sum_{n=1}^{\infty} |\langle f, f_n - g_n\rangle|^2 \;\le\; R\,\|f\|^2.$$

(We say then that $\{f_n - g_n\}_{n\in\mathbb{N}}$ is a *Bessel sequence* with *Bessel bound* R; the sequence $\{f_n - g_n\}_{n\in\mathbb{N}}$ need not have a positive lower frame bound).

Show that $\{g_n\}$ is a frame if $R < A$. Hint: use the triangle inequality in ℓ^2:

$$\left(\sum_{n=1}^{\infty} |c_n + d_n|^2\right)^{1/2} \;\le\; \left(\sum_{n=1}^{\infty} |c_n|^2\right)^{1/2} \;+\; \left(\sum_{n=1}^{\infty} |d_n|^2\right)^{1/2}.$$

(b) Show that if $\{h_n\}$ is a sequence in H which satisfies

$$R \;=\; \sum_{n} \|h_n\|^2 < \infty$$

then $\{h_n\}$ is a Bessel sequence with bound R.

(c) The exponentials $\{e^{2\pi i n T x}\}_{n\in\mathbb{Z}}$ are a frame in $L^2[-\frac{1}{2}, \frac{1}{2}]$ when $T < 1$. Use Problem 2, together with parts (a) and (b) of this problem, to formulate and prove a theorem establishing a sufficient condition on numbers $t_n \in \mathbb{R}$ so that $\{e^{2\pi i t_n x}\}_{n\in\mathbb{Z}}$ is a frame for $L^2[-\frac{1}{2}, \frac{1}{2}]$. Do you think your result is optimal?