A.8 Closed and Dense Sets

The smallest closed set that contains a given set is called its closure, defined precisely as follows.

**Definition A.57.** If $E$ is a subset of a topological space $X$, then the closure of $E$, denoted $\overline{E}$, is the smallest closed set in $X$ that contains $E$:

$$\overline{E} = \cap \{F \subseteq X : F \text{ is closed and } F \supseteq E\}.$$ 

If $\overline{E} = X$, then we say that $E$ is dense in $X$.

Often it is more convenient to use the following equivalent form of the closure of a set.

**Exercise A.58.** Given a subset $E$ of a topological space $X$, show that $\overline{E}$ is the union of $E$ and all the accumulation points of $E$.

There are many different notations and terminology that are commonly used when discussing subspaces of a normed space. In particular, some authors make the restriction that the term "subspace" is reserved to mean a "closed subspace." Other authors use the term "linear manifold" to denote a subspace that need not be closed. To avoid ambiguity, a subspace for us will mean a subspace in the usual vector space sense, i.e., a subset that is closed under both vector addition and scalar multiplication. We will refer to a subspace that is also a closed set as a closed subspace.

The typical method for showing that a subset of a metric space is dense is given in the next exercise.

**Exercise A.59.** Let $X$ be a metric space, and let $E \subseteq X$ be given. Show that $E$ is dense in $X$ if and only if for each $f \in X$ there exist a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $E$ such that $f_n \to f$.

In a finite-dimensional normed space, every subspace is a closed set. The following exercises illustrate that this need not be the case in infinite dimensions.

**Exercise A.60. (a)** Fix $1 \leq p \leq \infty$. Prove that

$$c_0^0 = \{x = (x_1, \ldots, x_N, 0, 0, \ldots) : N > 0, x_1, \ldots, x_N \in \mathbb{C}\}$$

is a subspace of $\ell^p$ that is not closed (with respect to the $\ell^p$-norm). Prove that $c_0^0$ is dense in $\ell^p(\mathbb{N})$ if $p < \infty$, but that it is not dense in $\ell^\infty$. The vectors in $c_0^0$ are sometimes called finite sequences because they contain at most finitely many nonzero components.

(b) Define

$$c_0 = \{x = (x_k)_{k=1}^\infty : \lim_{k \to \infty} x_k = 0\}.$$ 

Prove $c_0$ is a closed subspace of $\ell^\infty(\mathbb{N})$, and that $c_0$ is the closure of $c_0^0$ in the $\ell^\infty$-norm.
Exercise A.61. Show that the space $C_c(\mathbb{R})$ introduced in equation (A.3) is a dense subspace of $C_0(\mathbb{R})$ that is not closed (under the uniform norm).

The significance of closed subspaces is given in the following exercise.

Exercise A.62. Let $M$ be a subspace of a Banach space $X$. Then $M$ is itself a Banach space (using the norm inherited from $X$) if and only if $M$ is closed.

Hence, $c_0$ is a normed space that is not complete with respect to any norm $\| \cdot \|_p$, $1 \leq p \leq \infty$. Similarly, $C_c(\mathbb{R})$ is a normed space that is not complete with respect to the uniform norm (compare Exercise A.21).

We now introduce a definition that in some sense distinguishes between "small" and "large" infinite-dimensional spaces.

Definition A.63. A topological space that contains a countable dense subset is said to be separable.

Exercise A.64. (a) Show that if $I$ is a finite or countable index set, then $\ell^p(I)$ is separable for $1 \leq p < \infty$. Show that if $I$ is infinite, then $\ell^\infty(I)$ is not separable.

(b) Show that $C_0(\mathbb{R})$ is separable. ← A little tricky.

Additional Problems

A.19. Show that every finite-dimensional subspace of a normed linear space is closed.
A.58 Solution sketch. Let $A$ be the union of $E$ and the accumulation points of $E$, and suppose that $x \not\in A$. Then $x$ is not an accumulation point of $E$, so there exists an open neighborhood $U$ of $x$ such that $E \cap (U \setminus \{x\}) = \emptyset$. Since $x \not\in E$, this implies $U$ contains no points of $E$. Show that $U$ cannot contain any accumulation points of $E$ either, and conclude that $U \subseteq X \setminus A$. Therefore $X \setminus A$ is open, so $A$ is closed, and consequently $E \subseteq A$.

A.64 Hints: (b) Let $\theta_M$ be 1 on $[-M, M]$, zero outside $[-M - 1, M + 1]$, and linear on $[-M - 1, -M]$ and $[M, M + 1]$. Use the Weierstrass Approximation Theorem (Theorem A.77) to show that

\[ S = \left\{ \sum_{k=0}^{N} r_k x^k \theta_M(x) : M \in \mathbb{N}, N \geq 0, \Re(r_k), \Im(r_k) \in \mathbb{Q} \right\} \]

is countable and dense in $C_0(\mathbb{R})$. 
Problem

A.19 Solution sketch. Let $M$ be a finite-dimensional subspace of a normed space $X$. Suppose that $f_n \in M$ and $f_n \to g \in X$. If $g \notin M$, define

$$M_1 = M + \text{span}\{g\} = \{m + cg : m \in M, c \in \mathbb{C}\}.$$ 

Show that if $f = m + cg$ with $m \in M$ and $c \in \mathbb{C}$, then $\|f\|_{M_1} = \|m\| + |c|$ is a well-defined norm on $M_1$. By Theorem A.56, all norms on $M_1$ are equivalent, so $f_n \to g$ in the norm of $M_1$. But $\|g - f_n\|_{M_1} = \|f_n\| + 1 \geq 1$ for every $n$, so this is a contradiction.