\[ X^{**} \text{ and Reflexivity} \]

We know that if \( X \) is a normed space, then \( X^* \) is a Banach space. Hence \( X^{**} = (X^*)^* \), \( X^{***} = ((X^*)^*)^* \), etc., are all Banach spaces. If \( X \) is a Hilbert space, then \( X \cong X^* \) (and in fact the converse is also true, \( X \cong X^* \) implies \( X \) is a Hilbert space).

In this section we will explore the relation between \( X \) and \( X^{**} \) for general normed spaces.

Motivation
Choose \( \mu \in X^* \). Then, by definition, \( X \rightarrow \langle x, \mu \rangle \)

is a bounded linear functional on \( X \). But we could also fix \( x \in X \) and consider the mapping \( \mu \mapsto \langle x, \mu \rangle \). This is a mapping from \( X^* \) to \( \mathbb{F} \), and under our notational conventions, it is an antilinear functional on \( X^* \). This is one.
Thus each $x \in X$ determines an $\hat{x} \in X^{**}$. Not only that, but the mapping $X \rightarrow \hat{X}$ is an isometry, so there is a natural isomorphic embedding of $X$ into $X^{**}$. If this map is onto, then we will say that $X$ is reflexive.

$\mu \in X^*$

\[ x \in X \]

$\mu$ acts on all $\sigma$-elements $x \in X$, $x \mapsto \mu(x)$

$X \rightarrow \hat{X}$

$\mu \in X^*$

$X$ acts on the elements of $X^*$ $\mu \mapsto \mu(x)$

$X^{**}$
place the choice of notational convention is less convenient, so for this section, let us return to the standard functional notation \( \mu(x) \) instead of \( \langle x, \mu \rangle \).

We regard this notation as being linear in both \( x \) and \( \mu \).

So, again, we have that \( x \mapsto \mu(x) \) is a bounded linear functional on \( X \), but also if we fix \( x \in X \) and \( \mu \) then \( \mu \mapsto \mu(x) \) is a linear functional on \( X^* \). Since this functional is determined by \( x \), we call it \( \hat{x} \). That is:

\[
\begin{align*}
  x \in X & \implies \hat{x} : X^* \to Y \\
  \mu \mapsto \mu(x) & \text{ is a linear functional on } X^*.
\end{align*}
\]

Further, we’ll see that \( \hat{x} \) is bounded, so \( \hat{x} \in X^{**} \).
Theorem

Let $X$ be a normed linear space.

a. Given $x \in X$, define

$$\hat{x} : X^* \to \mathbb{F}$$

$$\mu \mapsto \hat{x}(\mu)$$

Then $\hat{x} \in X^{**}$, i.e., $\hat{x}$ is a bounded linear functional on $X^*$.

b. We have $\|\hat{x}\| = \|x\|$ for each $x \in X$.

Consequently,

$$T : X \to X^{**}$$

$$x \mapsto \hat{x}$$

is a linear isometry of $X$ into $X^{**}$.

Proof

a. Fix $x \in X$. Exercise: $\hat{x}$ is linear. For $\mu \in X^*$,

$$|\hat{x}(\mu)| = |\mu(x)| \leq \|\mu\| \cdot \|x\|.$$  

Therefore

This is $\|\mu\|_{X^*}$, the norm of $\mu$.  

\[ \| \hat{x} \|_{X^{**}} = \sup_{\| \mu \|_{X^*} = 1} | \hat{x}(\mu) | \]
\[ \leq \sup_{\| \mu \|_{X^*} = 1} \| \mu \|_{X^*} \| x \| \]
\[ = \| x \| \]

Thus \( \hat{x} \) is bounded, so \( \hat{x} \in X^{**} \).

b. Exercise: \( T \) is linear.

So, it remains only to show that \( \| \hat{x} \|_{X^{**}} = \| x \| \).

This follows from (according to) Hahn-Banach:
\[ \| \hat{x} \|_{X^{**}} = \sup_{\| \mu \|_{X^*} = 1} | \hat{x}(\mu) | \]
\[ = \sup_{\| \mu \|_{X^*} = 1} | \mu(x) | \]
\[ = \| x \| \text{ by Hahn-Banach.} \]

Thus \( T \) is an isometry. \( \square \)
Definition

If the map $T$ of the preceding theorem is surjective, then we say that $X$ is reflexive.

Thus, if $X$ is reflexive, then $T: X \rightarrow X^{**}$ is an isometric isomorphism, so $X \cong X^{**}$.

Note: Since $X^{**}$ is a Banach space, only Banach spaces can be reflexive.

Remark: $X \cong X^{**}$ does not imply that $X$ is reflexive—in order to be reflexive, the isomorphism must be given by the "natural map" $T$.

Notation

We call $T$ the natural map of $X$ into $X^{**}$. As usual, we identify $x \in X$ with its image $x \in X^{**}$ and therefore write $X \subset X^{**}$ in the sense of identification under the natural map. If $X$ is reflexive, then $X = X^{**}$, again in the sense of identification under the natural map.
Exercise
Fix $1 < p < \infty$. Show that $L^p(\mathbb{N})$ and $L^p(E)$, $E \subseteq \mathbb{R}$ Lebesgue measurable, are reflexive.

Note: It is not quite enough to say that we have shown that $L^p(E)^* \cong L^q(E)$, and therefore $L^p(E)^{**} \cong L^q(E)^* \cong L^{p'}(E) = L^p(E)$.

This is true, but to be reflexive the isomorphism $L^p(E) \rightarrow L^p(E)^{**}$ must be given by the natural map.

Exercise
Recall that $C_0^* \cong l^1$ and $(l^1)^* \cong l^\infty$, so $C_0^{**} \cong l^\infty$. Show that the natural map

$T: C_0 \rightarrow l^\infty$ is the inclusion map, i.e.,

$T_x = x$ for $x \in C_0$. Conclude that $C_0$ is not reflexive (why?)
Exercise (See Conway, p. 92-93)

Let $M$ be a closed subspace of a Banach space $X$.

Let $\rho_X : X \to X^{**}$ & $\rho_M : M \to M^{**}$ be natural maps.

Let $i : M \to X$ be an inclusion map, $i(x) = x$ for $x \in M$.

Show that isometry $\phi : M^{**} \to X^{**}$ s.t.

$$\rho_X \circ i = \phi \circ \rho_M.$$

Prove further that $\phi(M^{**}) = (M^\perp)^\perp$.

Exercise

Use the preceding exercise to show that if $X$ is reflexive, then any closed subspace of $X$ is reflexive.