Definition

\[ C_0 = \left\{ x = (x_k)_{k \in \mathbb{N}} : \lim_{k \to \infty} x_k = 0 \right\} \]

\[ C = \left\{ x = (x_k)_{k \in \mathbb{N}} : \lim_{k \to \infty} x_k \text{ exists} \right\} \]

Exercises

Let \( e_k = (0, \ldots, 0, 1, 0, 0, \ldots) \) with \( \uparrow k \text{th component} \) be the standard basis vectors and set

\[ e_0 = (1, 1, 1, \ldots) \]

a. Prove that \( \{ e_k \}_{k \in \mathbb{N}} \) is a Schauder basis for \( C_0 \). Specifically, show that if

\[ x = (x_k)_{k \in \mathbb{N}} \in C_0, \quad \text{then} \]

\[ x = \sum_{k=1}^{\infty} x_k e_k, \]

where \( \sum \) series converges in \( \ell^\infty \)-norm, and
Furthermore, this is the unique representation of \( x \) w.r.t. \( \{e_k\}_{k=0}^{\infty} \).

b. Prove that \( \{e_k\}_{k=0}^{\infty} \) is a Schauder basis for \( C \). Specifically, if \( x \in C \) and we set
\[
X_0 = \lim_{k \to \infty} X_k, \quad \text{and} \quad C_k = X_k - X_0, \quad k \in \mathbb{N},
\]
then
\[
X = \sum_{k=0}^{\infty} C_k e_k = X_0 e_0 + \sum_{k=1}^{\infty} (X_k - X_0) e_k
\]
converges in \( \ell^\infty \)-norm and is the unique representation of \( x \) w.r.t. \( \{e_k\}_{k=0}^{\infty} \).
Coordinates

With $e_0, e_k$ as before, set

$\mathcal{E} = \{e_k\}_{k \in \mathbb{N}}, \quad \mathcal{E}' = \{e_k\}_{k \geq 0},$

so that $\mathcal{E}$ is a Schauder basis for $c_0$ and $\mathcal{E}'$ is a
Schauder basis for $c$. Since $x = (x_k)_{k \in \mathbb{N}} \in c_0$
can be written in terms of the basis $\mathcal{E}$ as

$x = \sum_{k=1}^{\infty} x_k e_k,$

the coordinates of $x$ w.r.t. the
basis $\mathcal{E}$ are defined to be

$[x]_{\mathcal{E}} = (x_k)_{k \in \mathbb{N}} = (x_1, x_2, \ldots).$

On the other hand, if $x = (x_k)_{k \in \mathbb{N}} \in c_0$,

we express $x$ w.r.t. the basis $\mathcal{E}'$ we write

$x = \sum_{k=0}^{\infty} c_k e_k,$

where $c_0 = x_0 = \lim_{k \to \infty} x_k$

and $c_k = x_k - x_0$ for $k \in \mathbb{N}$. The coordinates

of $x$ w.r.t. $\mathcal{E}'$ are therefore defined to

be
\([X]_{E_1} = (c_k)_{k \geq 0} = (c_0, c_1, c_2, \ldots)\)
\[= (x_0, x_1 - x_0, x_2 - x_0, \ldots)\]

Note that in terms of coordinates w.r.t. \(E_1\),

\(c_0\) is described as

\[c_0 = \{ x \in C : [x]_{E_1} = (0, x_1, x_2, \ldots) \}\]

This gives us another intuitive explanation of why \(\text{dom}(c/c_0) = 1\).
Dual Spaces

The fact that $E$ & $E'$ are Schauder bases for $C_0$ & $C$, respectively, suggests how we may characterize their dual spaces.

Warning: Merely knowing that a Banach space $X$ has a particular Schauder basis $\{x_n\}_{n \in \mathbb{N}}$ is not sufficient to yield a characterization of $X^*$. Something extra is needed to obtain an explicit characterization. What extra do we know about these Schauder bases for $C_0$ & $C$?

**Exercise**

Given $y = (y_k)_{k \in \mathbb{N}} \in l^1$, define

$$
\mu_y : C_0 \to l^1,
$$

$$
\mu_y (\xi) = (\xi y_k)_{k \in \mathbb{N}} \mapsto \sum_{k=1}^{\infty} \xi_k \overline{y_k}.
$$

Show that $\mu_y \in C_0^*$, $\|\mu_y\| = \|y\|_1$, and

$$
T : l^1 \to C_0^*,
y \mapsto \mu_y
$$

is an antilinear isometric isomorphism of $l^1$ onto $C_0^*$. 
Using the fact that $E'$ is a Schauder basis for $C$, we can also characterize $C^*$. Surprisingly, we see that $C^* \cong C^*$ is isometrically isomorphic.

Rather than presenting $C^*$ as Theorem/proof, let us attempt to be somewhat Socratic in approach.

If $x \in C$, then for any $y \in C^*$ we know that

$$
\sum_{k=1}^{\infty} x_k y_k
$$

converges, since $C \subseteq l^\infty$. However, attempting to use $\mathbb{R}^C$ to identify all linear functionals on $C$ is not likely to work — here we are using the standard basis $E$ to express $x$ & $y$, while we know that $E$ is not a Schauder basis for $C$.

Instead, let us try to construct functionals based on the fact that $E' = \{e_k\}_{k \geq 0}$ is a Schauder basis for $C$. Given $x = (x_k)_{k \in \mathbb{N}}$, we should examine the coordinates of $x$ w.r.t. $E'$, namely,

$$
[x]_{E'} = (x_0, x_1 - x_0, x_2 - x_0, \ldots)
$$
\[ x_0 = \lim_{k \to \infty} x_k. \] Now, since the series
\[
x = x_0 e_0 + \sum_{k=1}^{\infty} (x_k - x_0) e_k
\]
converges in \( L^\infty \)-norm, if we fix \( \mu \in \mathcal{M} \), then the continuity of \( \mu \) implies that its action on \( x \) will be given by
\[
\langle x, \mu \rangle = x_0 \langle e_0, \mu \rangle + \sum_{k=1}^{\infty} (x_k - x_0) \langle e_k, \mu \rangle. \tag{*}
\]

**Exercise:** By considering \( x_n = (c_1, \ldots, c_N, 0, 0, 0, \ldots) \), show that
\[
\sum_{k=0}^{\infty} |\langle e_k, \mu \rangle| < \infty.
\]

Therefore, we can rewrite (*) as
\[
\langle x, \mu \rangle = x_0 \left( \langle e_0, \mu \rangle - \sum_{k=1}^{\infty} \langle e_k, \mu \rangle \right) + \sum_{k=1}^{\infty} x_k \langle e_k, \mu \rangle
\]
\[
= \sum_{k=0}^{\infty} x_k \overline{y}_k
\]
where
\[ y_0 = \langle e_0, \mu \rangle - \sum_{k=1}^{\infty} \langle e_k, \mu \rangle, \quad y_k = \langle e_k, \mu \rangle, \quad k \in \mathbb{N}. \]

Thus, we are led to associate \( \mu \) with the \( l' \) sequence

\[ y_\mu = (y_k)_{k=0}^{\infty} \quad \text{with } y_k \text{ as above.} \]

(Note \( y \) belongs to \( l' = l'(I) \)

where \( I = \{0,1,2,\ldots\} \))

**Theorem**

\[ T: \mathcal{C} \rightarrow l' \] is an antilinear isometric isomorphism.

\[ \mu \rightarrow y_\mu \]

**Proof**

The previous work shows that \( T \) is well-defined, and it is clearly antilinear.

For \( \mu \in \mathcal{C} \), let \( y_\mu \) be defined as above. If \( x \in \mathcal{C} \) then we have \( |x_0| \leq \|x\|_\infty \), so

\[ |\langle x, \mu \rangle| = \left| \sum_{k=0}^{\infty} x_k y_k \right| \leq \sum_{k=0}^{\infty} |x_k| |y_k| \leq |x_0| \sum_{k=0}^{\infty} |y_k| \]

Hence \( \|\mu\| \leq \sum_{k=0}^{\infty} \|y_k\| \).

To show equality, for each \( k \geq 0 \) let \( c_k \) be the scalar of unit modulus such that \( c_k y_k = y_k \). Consider the vectors...
\[ \chi_N = (c_1, \ldots, c_N, c_0, c_0, c_0, \ldots) \]
\[ = c_0 e_0 + \sum_{k=1}^{N-1} (c_k - c_0) e_k. \]

We have \( \chi_N \in C \) and \( \| \chi_N \|_\infty = 1. \)

\[ \langle \chi_N, \mu \rangle = c_0 \langle e_0, \mu \rangle + \sum_{k=1}^{N-1} (c_k - c_0) \langle e_k, \mu \rangle \]
\[ = c_0 \left( \langle e_0, \mu \rangle - \sum_{k=1}^{N-1} \langle e_k, \mu \rangle \right) + \sum_{k=1}^{N-1} c_k \langle e_k, \mu \rangle \]
\[ = c_0 \left( \langle e_0, \mu \rangle - \sum_{k=1}^{N-1} \langle e_k, \mu \rangle \right) + \sum_{k=1}^{N-1} |y_k| \]
\[ \rightarrow c_0 y_0 + \sum_{k=1}^{\infty} |y_k| \quad \text{as} \quad N \rightarrow \infty \]
\[ = |y_0| + \sum_{k=1}^{\infty} |y_k| \]
\[ = \| y_0 \|_1. \]

Hence we do have \( \| y_\mu \|_1 = \| y_0 \|_1. \)

Therefore it only remains to show that \( T \) is surjective.
Given yet, define \( \mu : c \rightarrow F \) by

\[
\langle x, \mu \rangle = \sum_{k=0}^{\infty} x_k \bar{y}_k = x_0 \bar{y}_0 + \sum_{k=1}^{\infty} x_k \bar{y}_k
\]

Exercise: Verify that \( \mu \in c^* \).

Then for \( m \in N \) we have

\[
\langle e_m, \mu \rangle = x_0 \bar{y}_0 + \sum_{k=0}^{\infty} \delta_{mk} \bar{y}_k = \bar{y}_m.
\]

For \( m = 0 \), we have

\[
\langle e_0, \mu \rangle = \bar{y}_0 + \sum_{k=1}^{\infty} \bar{y}_k = \bar{y}_0 + \sum_{k=1}^{\infty} \langle e_k, \mu \rangle,
\]

so

\[
\bar{y}_0 = \langle e_0, \mu \rangle - \sum_{k=1}^{\infty} \langle e_k, \mu \rangle.
\]

Thus \( y_\mu = (y_k)_{k=0}^\infty = \bar{y}_0 \), so \( T \) is surjective.
Thus, we've shown that $C^*$ is isometrically isomorphic to $l^1$. We did it by deducing what $l^1$ sequence we should associate with $\mu \in C^*$. This is counter to my first inclination, which is to try to map $l^1$ to $C^*$. And now we can see exactly how to do this. Use the same notation as before, i.e., given $x = (x_k)_{k \in \mathbb{N}} \in C$, let $x_0 = \lim_{k \to \infty} x_k$.

While vectors in $C$ are indexed by $\mathbb{N}$, let us use $N_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ for $l^1$ index set for $l^1$. Given $y = (y_0)_{k=0}^{\infty} \in l^1 = l^1(\mathbb{N}_0)$ define

$$
\mu_y : C \rightarrow l^1 \rightarrow \sum_{k=0}^{\infty} x_k y_k.
$$

**Exercise:**
Show directly that $T : l^1 \rightarrow C^*$

$$
T(y) = \mu_y
$$

is an antilinear isometric isomorphism.
Final Thought

In summary, while $C_0$ is a closed subspace of $C$ and in fact is a hyperplane in $C$, these two spaces have isometric dual spaces. This leads to the question:

Are $C_0$ and $C$ isometrically isomorphic?

Acknowledgments

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