FUNCTIONAL ANALYSIS LECTURE NOTES:
WEAK AND WEAK* CONVERGENCE

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1. Weak and Weak* Convergence of Vectors

Definition 1.1. Let $X$ be a normed linear space, and let $x_n, x \in X$.

a. We say that $x_n$ converges, converges strongly, or converges in norm to $x$, and write $x_n \rightarrow x$, if
\[ \lim_{n \to \infty} \|x - x_n\| = 0. \]

b. We say that $x_n$ converges weakly to $x$, and write $x_n \overset{w}{\rightarrow} x$, if
\[ \forall \mu \in X^*, \lim_{n \to \infty} \langle x_n, \mu \rangle = \langle x, \mu \rangle. \]

Exercise 1.2. a. Show that strong convergence implies weak convergence.

b. Show that weak convergence does not imply strong convergence in general (look for a Hilbert space counterexample).

If our space is itself the dual space of another space, then there is an additional mode of convergence that we can consider, as follows.

Definition 1.3. Let $X$ be a normed linear space, and suppose that $\mu_n, \mu \in X^*$. Then we say that $\mu_n$ converges weak* to $\mu$, and write $\mu_n \overset{w^*}{\rightarrow} \mu$, if
\[ \forall x \in X, \lim_{n \to \infty} \langle x, \mu_n \rangle = \langle x, \mu \rangle. \]

Note that weak* convergence is just “pointwise convergence” of the operators $\mu_n$!

Remark 1.4. Weak* convergence only makes sense for a sequence that lies in a dual space $X^*$. However, if we do have a sequence $\{\mu_n\}_{n \in \mathbb{N}}$ in $X^*$, then we can consider three types of convergence of $\mu_n$ to $\mu$: strong, weak, and weak*. By definition, these are:
\[ \mu_n \rightarrow \mu \iff \lim_{n \to \infty} \|\mu - \mu_n\| = 0, \]
\[ \mu_n \overset{w}{\rightarrow} \mu \iff \forall T \in X^{**}, \lim_{n \to \infty} \langle \mu_n, T \rangle = \langle \mu, T \rangle, \]
\[ \mu_n \overset{w^*}{\rightarrow} \mu \iff \forall x \in X, \lim_{n \to \infty} \langle x, \mu_n \rangle = \langle x, \mu \rangle. \]
Exercise 1.5. Given \( \mu_n, \mu \in X^* \), show that

\[
\mu_n \rightarrow \mu \quad \Rightarrow \quad \mu_n \overset{w}{\rightarrow} \mu \quad \Rightarrow \quad \mu_n \overset{w^*}{\rightarrow} \mu.
\]  
(1.1)

If \( X \) is reflexive, show that

\[
\mu_n \overset{w}{\rightarrow} \mu \quad \iff \quad \mu_n \overset{w^*}{\rightarrow} \mu.
\]

In general, however, the implications in (1.1) do not hold in the reverse direction.

Lemma 1.6. a. Weak* limits are unique.

b. Weak limits are unique.

Proof. Suppose that \( X \) is a normed linear space, and that we had both \( \mu_n \overset{w^*}{\rightarrow} \mu \) and \( \mu_n \overset{w^*}{\rightarrow} \nu \) in \( X^* \). Then, by definition,

\[ \forall x \in X, \quad \langle x, \mu \rangle = \lim_{n \to \infty} \langle x, \mu_n \rangle = \langle x, \nu \rangle, \]

so \( \mu = \nu \).

b. Suppose that we have both \( x_n \overset{w}{\rightarrow} x \) and \( x_n \overset{w}{\rightarrow} y \) in \( X \). Then, by definition,

\[ \forall \mu \in X^*, \quad \langle x, \mu \rangle = \lim_{n \to \infty} \langle x_n, \mu \rangle = \langle y, \mu \rangle. \]

Hence, by Hahn–Banach,

\[ \|x - y\| = \sup_{\|\mu\| = 1} |\langle x - y, \mu \rangle| = 0, \]

so \( x = y \). \qed

It is trivial to show that strongly convergent sequences are bounded. However, we need some fairly sophisticated machinery (the Uniform Boundedness Principle) to show that weakly convergent and weak* convergent sequences are likewise bounded.

Exercise 1.7. a. Show that weak* convergent sequences in the dual of a Banach space are bounded.

Give an example of an unbounded but weak* convergence sequence in the dual of an incomplete normed space.

Hint: The dual space of \( c_{00} \) under the \( \ell^\infty \) norm is \((c_{00})^* \cong \ell^1\).

b. Show that weakly convergent sequences in a normed space are bounded.

Next, we will show that strong convergence is equivalent to weak convergence in finite-dimensional spaces.

Lemma 1.8. If \( X \) is a finite-dimensional vector space, then strong convergence is equivalent to weak convergence.
Proof. Consider first the case that $X = \mathbb{F}^d$ under the Euclidean norm $\| \cdot \|_2$. Suppose that $x_n \xrightarrow{w} x$ in $\mathbb{F}^d$. Then for each standard basis vector $e_k$, we have

$$x_n \cdot e_k \to x \cdot e_k, \quad k = 1, \ldots, d.$$ 

That is, weak convergence implies componentwise convergence. But since there are only finitely many components, this implies norm convergence, since

$$\|x - x_n\|_2^2 = \sum_{k=1}^{d} |x \cdot e_k - x_n \cdot e_k|^2 \to 0 \quad \text{as } n \to \infty.$$ 

For the general case, choose any basis $B = \{e_1, \ldots, e_d\}$ for $X$, and use the fact that all norms on $X$ are equivalent to define an isomorphism between $X$ and $\mathbb{F}^d$. \hfill \Box

Often, there exists a connection between componentwise or pointwise convergence and weak convergence. This is related to the question of whether “point evaluation” are continuous linear functionals on a given space.

**Example 1.9.** Fix $1 \leq p \leq \infty$, and consider the space $\ell^p$. As usual, given $x \in \ell^p$ and $y \in \ell^p$, write

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k \bar{y}_k.$$ 

The standard basis vectors $e_k$ belong to every $\ell^p$, and hence $e_k \in \ell^{p'} \subseteq (\ell^p)^*$ (with equality if $p < \infty$). Therefore, if we have $x_n = (x_n(k))_{k \in \mathbb{N}}$ and $y = (y(k))_{k \in \mathbb{N}}$ and we know that $x_n \xrightarrow{w} y$, then we have for each $k \in \mathbb{N}$ that

$$x_n(k) = \langle x_n, e_k \rangle \to \langle y, e_k \rangle = y(k).$$

Thus,

$$\text{weak convergence in } \ell^p \implies \text{componentwise convergence in } \ell^p.$$ 

The converse is not true in general, but the following result gives necessary and sufficient conditions, at least for some $p$.

**Exercise 1.10.** Fix $1 < p < \infty$, and let $x_n, y \in \ell^p$ be given. Prove that the following two statements are equivalent.

a. $x_n \xrightarrow{w} y$.

b. $x_n(k) \to y(k)$ for each $k$ (componentwise convergence) and $\sup \|x_n\|_p < \infty$.

What happens if $p = 1$ or $p = \infty$?

**Exercise 1.11.** Suppose that $x_n, y \in \ell^1$ are given. Since $\ell^1 \cong c_0$, we can consider weak* convergence of $x_n$ to $y$. Prove that the following two statements are equivalent.

a. $x_n \xrightarrow{w^*} y$. 
b. \( x_n(k) \to y(k) \) for each \( k \) (componentwise convergence) and \( \sup \| x_n \|_1 < \infty \).

**Exercise 1.12.** Let \( f_n, f \in C_0(\mathbb{R}) \) be given. Since \( C_0(\mathbb{R})^* \cong M_b(\mathbb{R}) \), we can consider weak convergence of \( f_n \) to \( f \). Prove that the following two statements are equivalent.

a. \( f_n \overset{w}{\to} f \).

b. \( f_n(x) \to f(x) \) pointwise for each \( x \), and \( \sup_k f_n(k) < 1 \).

**Exercise 1.13.** Let \( \mu_n, \mu \in M_b(\mathbb{R}) \) be given. Show that \( \mu_n \overset{w^*}{\to} \mu \) does not imply \( \| \mu_n \| \to \| \mu \| \).

**Exercise 1.14.** Fix \( 1 < p < \infty \), and let \( f_n \in L^p(\mathbb{R}) \) be given. Prove that the following two statements are equivalent.

a. \( f_n \overset{w}{\to} 0 \).

b. \( \int_E f_n \to 0 \) for every \( E \subseteq \mathbb{R} \) with \( |E| < \infty \), and \( \sup \| f_n \|_p < \infty \).

**Proof.** \( b \Rightarrow a \). Suppose that statement \( b \) holds, and let \( R = \sup \| f_n \|_p \). Choose any \( g \in L^{p'}(\mathbb{R}) \). Since the step functions are dense in \( L^{p'}(\mathbb{R}) \), we can find a function of the norm

\[
\varphi = \sum_{k=1}^{M} c_k \chi_{F_k},
\]

with each \( F_k \) a measurable subset of \( \mathbb{R} \), such that

\[
\| g - \varphi \|_{p'} < \frac{\varepsilon}{4R}.
\]

Since \( \varphi \in L^{p'}(\mathbb{R}) \), there exists a compact set \( K \) such that if we set \( \psi = \varphi \chi_K \), then we have

\[
\| \varphi - \psi \|_{p'} < \frac{\varepsilon}{4R}.
\]

Furthermore, note that \( \psi \) is a step function, since

\[
\psi = \sum_{k=1}^{M} c_k \chi_{F_k} \chi_K = \sum_{k=1}^{M} c_k \chi_{E_k},
\]

where \( E_k = F_k \cap K \).

Set

\[
C = \sum_{k=1}^{M} |c_k|,
\]

and assume for now that \( C > 0 \). Since each \( E_k \) has finite measure, by hypothesis we can find an integer \( N \) such that

\[
n > N \implies \left| \int_{E_k} f_n \right| < \frac{\varepsilon}{2C}.
\]
Hence for \( n > N \) we have

\[
|\langle g, f_n \rangle| \leq |\langle g - \varphi, f_n \rangle| + |\langle \varphi - \psi, f_n \rangle| + |\langle \psi, f_n \rangle| \\
< \|g - \varphi\| \|f_n\| + \|\varphi - \psi\| \|f_n\| + \left| \int_{E_k} \sum_{k=1}^M c_k \chi_{E_k} f_n \right| \\
< \frac{\varepsilon}{4R} R + \frac{\varepsilon}{4R} R + \sum_{k=1}^M |c_k| \left| \int_{E_k} f_n \right| \\
< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \sum_{k=1}^M |c_k| \frac{\varepsilon}{2C} \\
= \varepsilon.
\]

If \( C = 0 \) then we still obtain \( |\langle g, f_n \rangle| < \varepsilon \). This shows that

\[
0 \leq \limsup_{n \to \infty} |\langle g, f_n \rangle| \leq \varepsilon.
\]

Since this is true for every \( \varepsilon \), we conclude that

\[
\lim_{n \to \infty} |\langle g, f_n \rangle| = 0.
\]

And since this is true for every \( g \in L^p(\mathbb{R}) \), we have \( f_n \overset{w}{\to} 0 \). \( \square \)

**Exercise 1.15.** Let \( H \) be a Hilbert space. Show that

\[
f_n \to f \iff f_n \overset{w}{\to} f \text{ and } \|f_n\| \to \|f\|.
\]

**Exercise 1.16.** Let \( H \) and \( K \) be Hilbert spaces, and let \( T \in \mathcal{B}(H, K) \) be a compact operator. Show that

\[
f_n \overset{w}{\to} f \implies Tf_n \to Tf.
\]

Thus, a compact operator maps weakly convergent sequences to strongly convergent sequences.
2. Convergence of Operators

We can apply similar notions to convergence of operators.

**Definition 2.1.** Let $X$, $Y$ be normed linear spaces, and let $A_n, A \in B(X,Y)$ be given.

a. We say that $A_n$ **converges in operator norm to** $A$, or that $A_n$ **is uniformly operator convergent to** $A$, and write $A_n \to A$, if

$$\lim_{n \to \infty} \|A - A_n\| = 0.$$ 

Rewriting the definition of operator norm, this is equivalent to

$$\lim_{n \to \infty} \left( \sup_{\|x\|=1} \|Ax - A_nx\| \right) = 0.$$ 

b. We say that $A_n$ **converges in the strong operator topology** (SOT) to $A$, or that $A_n$ **is strongly operator convergent to** $A$, if

$$\forall x \in X, \; A_nx \to Ax \text{ (strong convergence in } Y).$$

Equivalently, this holds if

$$\forall x \in X, \; \lim_{n \to \infty} \|Ax - A_nx\| = 0.$$ 

c. We say that $A_n$ **is weakly operator convergent to** $A$, if

$$\forall x \in X, \; A_nx \xrightarrow{w} Ax \text{ (weak convergence in } Y).$$

Equivalently, this holds if

$$\forall x \in X, \; \forall \mu \in Y^*, \; \lim_{n \to \infty} \langle A_nx, \mu \rangle = \langle Ax, \mu \rangle.$$ 

**Remark 2.2.** In particular, consider the case $Y = \mathbb{F}$, i.e., the operators $A_n$ are bounded linear functionals on $X$. Since $Y = Y^*$, strong and weak convergence in $Y$ are equivalent. Hence for this case, strong operator convergence and weak operator convergence are equivalent, and in fact, they are simply weak* convergence of the operators $A_n$ in $X^*$. Further, uniform operator convergence is simply operator norm convergence of the operators $A_n$ in $X^*$. 