1. Prove that our definition of the Fourier transform on $L^2(\mathbb{R})$ is well-defined, i.e., it is independent of the choice of sequence $\{f_n\}_{n \in \mathbb{N}}$.

2. Suppose that $f \in L^1(\mathbb{R})$ is such that $\hat{f} \in L^2(\mathbb{R})$. Prove that $f \in L^2(\mathbb{R})$.

3. Let $H$ be a Hilbert space. A continuous linear operator $A: H \to H$ is Hilbert–Schmidt if there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ for $H$ such that
$$\|A\|_{\text{HS}}^2 = \sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty.$$  

a. Prove that $\|A\|_{\text{HS}}$ does not depend on the choice of orthonormal basis, i.e., if $\{f_n\}_{n \in \mathbb{N}}$ is another orthonormal basis for $H$, then $\sum_{n=1}^{\infty} \|Af_n\|^2 = \sum_{n=1}^{\infty} \|Ae_n\|^2$.

b. Prove that the Hilbert–Schmidt norm dominates the operator norm, i.e., $\|A\| \leq \|A\|_{\text{HS}}$.

c. Suppose that $A$ is a compact, self-adjoint operator. Then by the Spectral Theorem, there exists an orthonormal basis $\{e_n\}$ for $\text{range}(A)$ consisting of eigenvectors of $A$, and corresponding nonzero real eigenvalues $\lambda_n$, such that $Af = \sum \lambda_n \langle f, e_n \rangle e_n$. Prove that $A$ is Hilbert–Schmidt if and only if $\sum |\lambda_n|^2 < \infty$, and in this case $\|A\|_{\text{HS}}^2 = \sum |\lambda_n|^2$.

d. Suppose that $A$ is an integral operator on $L^2(\mathbb{R})$ with kernel $k$, i.e.,
$$Af(x) = \int k(x, y) f(y) \, dy.$$  
Prove that if $k \in L^2(\mathbb{R})^2$, then $A$ is Hilbert–Schmidt and $\|A\|_{\text{HS}} = \|k\|_2$.

4. a. Let $A, B$ be self-adjoint but possibly unbounded operators on a Hilbert space $H$. Prove that if $f \in \text{dom}(AB) \cap \text{dom}(BA)$, then
$$\|Af\| \|Bf\| \geq \frac{1}{2} \left| \langle [A, B]f, f \rangle \right|,$$  
where $[A, B] = AB - BA$ is the commutator of $A$ and $B$.

b. Show that equality holds in part a if and only if $Af = icBf$ for some $c \in \mathbb{R}$.

c. Apply part a to the position and momentum operators $Pf(x) = xf(x)$ and $Mf(x) = \frac{1}{2\pi} f'(x)$ to derive the Classical Uncertainty Principle (for the case $x_0 = \xi_0 = 0$).