3.2 Distributions

Distributions, or generalized functions, are continuous linear functionals on the following spaces:

\[ C^\infty_0(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{C} : f \text{ is infinitely differentiable and compactly supported} \} \]

\[ \mathcal{S}(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{C} : f \text{ is infinitely differentiable and } x^m f^{(n)}(x) \in L^\infty(\mathbb{R}) \quad \forall m,n \geq 0 \} \]

\[ C^\infty(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{C} : f \text{ is infinitely differentiable} \} \]

Note: We are not requiring functions in \( C^\infty(\mathbb{R}) \) to be bounded (or their derivatives), unlike the space \( C^\infty_0(\mathbb{R}) \).

These spaces are topological vector spaces, whose topologies are determined by a family of seminorms. Their dual spaces are spaces of distributions.

* See Appendix for more details on topological vector spaces.
Since
\[ C_c(\mathbb{R}) \quad \text{smallest domain} \quad \text{imposes least restrictions on continuity of a linear functional} \]
\[ \bigcap \]
\[ D(\mathbb{R}) \quad \text{largest domain} \quad \text{imposes greatest restrictions on continuity} \]

if there is some connection between the topologies on these spaces then we can hope that
\[ C_c(\mathbb{R})^* \overset{\text{def}}{=} \text{space of distributions } \mathcal{D}'(\mathbb{R}) \]
\[ \bigcup \]
\[ D(\mathbb{R})^* \overset{\text{def}}{=} \text{space of tempered distributions } \mathcal{S}'(\mathbb{R}) \]
\[ \bigcup \]
\[ C^\infty(\mathbb{R})^* \overset{\text{def}}{=} \text{space of compactly supported distributions } \mathcal{E}'(\mathbb{R}) \]

We will see that these inclusions are correct.

Unfortunately, these spaces are not Banach spaces, so we must first review the definitions of their topologies, or equivalently, the definition of convergence in these spaces (we did this for \( L^1(\mathbb{R}) \) in an earlier section).
Convergence in $C^\omega(\mathbb{R})$ is defined as follows.

**Definition**

Given $f_k, g \in C^\omega(\mathbb{R})$, we say that $f_k \rightarrow g$ in $C^\omega(\mathbb{R})$ if

a. $f_k \in C^\omega(\mathbb{R})$ s.t. $\text{supp}(f_k) \subseteq K$, $\forall N, \delta$

b. $\forall N > 0, \lim_{K \rightarrow 0} \| g^{(n)} - f_k^{(n)} \|_\infty = 0$.

**Remark**

A basic philosophy is that "topology is equivalent to a convergence criterion." Above, we give a convergence criterion, & there is an equivalent formulation in terms of a topology. Unfortunately, the equivalence is more complicated for $C^\omega(\mathbb{R})$ than it is for $L^1(\mathbb{R})$ or $C^0(\mathbb{R})$. The topology on $L^1(\mathbb{R})$ is induced by the countable family of seminorms

$$p_{mn}(f) = \| x^m f^{(n)}(x) \|_\infty, \quad m, n > 0.$$  

The corresponding convergence criterion is

$$f_k \rightarrow g \text{ in } L^1(\mathbb{R}) \text{ if } \forall m, n > 0, \ p_{mn}(f_k - g) \rightarrow 0.$$  

Because there are countably many seminorms for $L^1(\mathbb{R})$, & because the topology is Hausdorff, i.e.,
\[ \rho_{mn}(f) = 0 \quad \forall m, n \geq 0 \implies f = 0 \]

The Schwartz space \( S(\mathbb{R}) \) is metrizable, leading to an easy connection between topology & convergence.

Likewise \( C^\infty(\mathbb{R}) \) is determined by a countable family of seminorms, namely

\[ \rho_{K,n}(f) = \| f^{(n)} \|_{\infty}, \quad n \geq 0, K \text{ compact.} \]

WLOG we can reduce \( \rho_{K,n} \) to a countable family

\[ \rho_{mn}(f) = \| f^{(m)} \|_{[-m,m]} \|_{\infty}, \quad m, n \geq 0, \]

and \( C^\infty(\mathbb{R}) \) is metrizable as well.

The situation for \( C^\infty_c(\mathbb{R}) \) is more complicated. In technical terms, \( C^\infty_c(\mathbb{R}) \) is the inductive limit of the spaces

\[ C^\infty([-m,m]) = \{ f \in C^\infty(\mathbb{R}) : \text{supp}(f) \subseteq [-m,m] \}. \]

Each individual space \( C^\infty([-m,m]) \) is determined by a countable family of seminorms.
\[ \rho_m(f) = \| f^m \|_{X[-m,m]} \|_\infty \quad n \geq 0 \]

and is metrizable. The inductive limit of these topologies is roughly determined by the requirement that a function on \( C^\infty(M) \) be continuous if and only if each restriction to \( C^\infty([-m,m]) \) be continuous. This leads to the definition of convergence given above.

These issues, and in particular the equivalence between topology and convergence, is presented in detail in the Appendix. Here we will simply take the convergence criterion given above as a definition.

The remainder of this section will focus on \( C^\infty(M) \) and its dual space \( D'(M) \), later sections will consider \( \mathcal{S}(\mathbb{R}), C^\infty(\mathbb{R}), \) and their duals.
We will concentrate in this section on the space of distributions $\mathcal{D}'(\mathbb{R})$, see the appendix for expanded discussion of these definitions.

**Definition**

Given $f_n, g \in C_c^\infty(\mathbb{R})$, we say that $f_n \to g$ in $C_c^\infty(\mathbb{R})$ if

1. $f_n \to 0$ uniformly on compact sets.
2. $\lim_{n \to \infty} \| f_n \|_{C_c^k(\mathbb{R})} = 0$, for all $k \geq 0$.

**Definition**

1. A linear functional $T : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ is **continuous** if $f_n \to 0$ in $C_c^\infty(\mathbb{R}) \implies \langle f_n, T \rangle \to 0$.

2. The space of **distributions** on $\mathbb{R}$ is

$$\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^* = \{ T : C_c^\infty(\mathbb{R}) \to \mathbb{C} : T \text{ is a continuous linear functional} \}$$

3. A distribution $T \in \mathcal{D}'(\mathbb{R})$ is **positive**, denoted $T \geq 0$, if

$$\forall f \in C_c^\infty(\mathbb{R}), \quad f \geq 0 \implies \langle f, T \rangle \geq 0$$

**Exercise:**

$\mathcal{D}'(\mathbb{R})$ is a vector space.
Exercise

Let \( k \in C_c^\infty(\mathbb{R}) \) be s.t. \( \int k = 1 \), and set \( k_\lambda(x) = \lambda k(\lambda x) \).

Prove that

\[ \forall f \in C_c^\infty(\mathbb{R}), \quad f * k_\lambda \to f \quad \text{in} \quad C_c^\infty(\mathbb{R}), \quad \text{as} \quad \lambda \to \infty. \]

Conclude that if \( \mu \in \mathcal{D}'(\mathbb{R}) \), then

\[ \langle f * k_\lambda, \mu \rangle \to \langle f, \mu \rangle \quad \text{as} \quad \lambda \to \infty. \]

Exercise

Define \( \delta : C_c^\infty(\mathbb{R}) \to \mathbb{C} \) by

\[ \langle f, \delta \rangle = \int f(x) \, dx, \quad f \in C_c^\infty(\mathbb{R}). \]

Prove that \( \delta \in \mathcal{D}'(\mathbb{R}) \), & \( \delta \) is a positive distribution \( (\delta \geq 0) \).

Exercise

Prove that differentiation is a continuous operation on \( C_c^\infty(\mathbb{R}) \), i.e.,

\[ f_k \to f \quad \text{in} \quad C_c^\infty(\mathbb{R}) \quad \Rightarrow \quad f'_k \to f' \quad \text{in} \quad C^\infty(\mathbb{R}). \]
Exercise: Products of distributions & smooth functions.

Given $T \in \mathcal{D}'(\mathbb{R})$ & $\psi \in C^\infty(\mathbb{R})$, define $T\psi: C^\infty_c(\mathbb{R}) \to \mathbb{C}$ by

$$\langle f, T\psi \rangle = \langle f \psi, T \rangle, \quad f \in C^\infty_c(\mathbb{R}).$$

Show that $T\psi \in \mathcal{D}'(\mathbb{R})$.

Remark: Unfortunately, we cannot define the product of two arbitrary distributions - some restrictions are required.

However, the process of "imitating" functional definitions for distributions is often valid & leads to extensions of ideas to & setting of distributions.

Exercise

Define the translation $T_{\mu}$, modulation $M\mu$, & dilation $D_\mu$ of a distribution $\mu$, & prove they are distributions.

Explain why $\langle f, \overline{\mu} \rangle = \overline{\langle f, \mu \rangle}$ is an appropriate definition of the complex conjugate of a distribution, & prove $\overline{\mu} \in \mathcal{D}'(\mathbb{R})$. 
Motivation

If \( g \in L^1_{loc}(\mathbb{R}) \) & \( f \in C_0^\infty(\mathbb{R}) \), then \( \tilde{f}(x) = \overline{f(-x)} \)

\[
(f * g)(x) = \int f(x-y) g(y) \, dy = \int \tilde{f}(y-x) \overline{g(y)} \, dy = \langle T_x \tilde{f}, g \rangle.
\]

We use this formula to extend to convolutions with distributions.

Definition: Convolution

Given \( \mu \in \mathcal{D}'(\mathbb{R}) \) & \( f \in C_0^\infty(\mathbb{R}) \), we define \( f * \mu \) to be the function given by

\[
(f * \mu)(x) = \langle T_x \tilde{f}, \mu \rangle
\]

Example

\[
(f * \delta)(x) = \langle T_x \tilde{f}, \delta \rangle = \overline{T_x \tilde{f}(0)} = f(x).
\]

Thus \( f * \delta = f \), i.e., \( \delta \) is an identity for convolution. (at least on \( C_0^\infty(\mathbb{R}) \)).
Exercise
Show that \((f * \delta')(x) = -f'(x)\).

Exercise
Show that convolution commutes with translation, i.e., if \(f \in C^\infty(\mathbb{R})\), \(\mu \in \mathcal{D}'(\mathbb{R})\), and \(a \in \mathbb{R}\), then
\[ T_a (f * \mu) = T_a f * \mu = f * T_a \mu. \]

*Note: \(\delta'\) is a distribution whose rule is
\[ \langle f, \delta' \rangle = -f'(0), \quad f \in C^\infty(\mathbb{R}). \]
Exercise
Let \( \mu \in \mathcal{D}'(\mathbb{R}) \) & \( f \in C_c^\infty(\mathbb{R}) \) be given.

a. Show \( f * \mu \in C(\mathbb{R}) \).

Hint: Show \( T_a f \to f \) on \( C_c^\infty(\mathbb{R}) \) as \( a \to 0 \).

b. Show \( f * \mu \in C^1(\mathbb{R}) \), & \( (f * \mu)' = f' * \mu \).

Hint: Show \( \frac{T_a f - f}{a} \to f' \) in \( C_c^\infty(\mathbb{R}) \) as \( a \to 0 \).

c. Show \( f * \mu \in C^\infty(\mathbb{R}) \).

d. Prove that \( \forall x \in \mathbb{R} \),

\[
\langle f, \mu \rangle = (f * \mu)(x) - \int_0^1 (f * \mu)'(xy) \, dy
\]

Hint: Simply apply the Fundamental Theorem of Calculus.

Exercise
Let \( \delta_t = T_t \delta \), so \( \langle f, \delta_t \rangle = f(t) \) for \( f \in C_c^\infty(\mathbb{R}) \).

Find a formula for \( f * \delta_t \).
The following result is a $\mathbb{R}$ analogue of "continuity = boundedness" for linear functions on a normed space. An expanded version of this result is given in $\mathbb{R}$ Appendix. We use $\mathbb{R}$ notation
\[ \|f\|_N = \max_{n=0,\ldots,N} \|f^{(n)}\|_{\infty}. \]

**Theorem**

If $\mu : C_{c}^{\infty}(\mathbb{R}) \to \mathbb{C}$ is a linear functional, $\mu$ is continuous,

1. $\mu$ is continuous, i.e., $\mu \in \mathbb{D}'(\mathbb{R})$.
2. For compact $K \subseteq \mathbb{R}$, $\exists N_k > 0, \forall C_k > 0$ s.t.
   \[ f \in C_{c}^{\infty}(K) \implies |< f, \mu >| \leq C_k \|f\|_N. \]

**Proof:**

a) $\Rightarrow$ b. Suppose $\mu \in \mathbb{D}'(\mathbb{R})$, & fix any compact $K \subseteq \mathbb{R}$. Suppose that $\forall$ any $C, N$ s.t.

\[ f \in C_{c}^{\infty}(K) \implies |< f, \mu >| \leq C \|f\|_N. \]

Then for each $C = N = k \in 1/N$ we can find $f_k \in C_{c}^{\infty}(K)$ s.t.

\[ |< f_k, \mu >| > k \|f_k\|_N. \]

In particular, $|< f_k, \mu >| > 0$, so we can define
\[ \Phi_k = \frac{f_k}{\langle f_k, \mu \rangle}. \]

Then we have

\[ 1 = \langle \Phi_k, \mu \rangle \geq k \| \Phi_k \|_k, \]

so \[ \| \Phi_k \|_k \leq \frac{1}{k}. \] Hence \[ \Phi_k \to 0 \quad \text{in } C^0(\mathbb{R}), \]

but this contradicts the fact that \( \mu \) is continuous, which implies \[ \langle \Phi_k, \mu \rangle \to 0. \]

**Definition**

If a single \( N \) will work for all compact \( K \)

(both possibly different \( C_K \) for smallest such \( N \))

is called the order of \( T \).

**Exercise**

Show that \( f \) has finite order.

**Exercise**

Show that \[ \langle f, \mu \rangle = \sum_{n=0}^{\infty} f^{(n)}(n) \] defines a distribution, & that \( \mu \) has infinite order.
Exercise

Let \( \mu = \sum_{n \in \mathbb{Z}} \delta_n \), i.e.,

\[
\forall f \in C_c^\infty(\mathbb{R}), \quad \langle f, \mu \rangle = \sum_{n \in \mathbb{Z}} f(n).
\]

Pictorially, we may (with some poetic license) imagine that \( \mu \) "looks" like this:

```
\[\begin{array}{c}
-3 -2 -1 0 1 2 \\
\end{array}\]
```

This picture inspires many names for \( \mu \), including:

- \( \mathbb{Z} \) delta train or train of deltas (or Diracs)
- \( \mathbb{Z} \) Dirac comb
- \( \mathbb{Z} \) Shah distribution, because the picture is suggestive of \( \mathbb{Z} \) Cyrillic letter "sha", which is written \( \text{Ш} \). For this reason, \( \mu \) is sometimes denoted by \( \text{Ш} \) symbol \( \text{Ш} \).

a. Prove that \( \mu \in \mathcal{D}'(\mathbb{R}) \).

b. Prove that \( \mu \) has order 0, but the constant \( C \) cannot be chosen independent of a compact set \( K \).
Definition: Convergence of Distributions

If $\mu, \nu \in \mathcal{D}'(\mathbb{R})$, then we say that $\mu_n \to \mu$ in $\mathcal{D}'(\mathbb{R})$ if

$$\forall f \in C^\infty_c(\mathbb{R}), \quad \langle f, \mu_n \rangle \to \langle f, \mu \rangle.$$

That is, convergence of distributions is convergence in the weak* topology.

Remark: This is a "natural" to put on a dual space in many respects - the elements of a dual space are functionals on $C^\infty_c(\mathbb{R})$, and weak* convergence is simply "pointwise convergence" of these functions (with $\mathbb{R}$ "points" being the elements of $C^\infty_c(\mathbb{R})$).

Exercise

Let $\theta \in C^\infty_c(\mathbb{R})$ be s.t. $\theta = 1$ in a neighborhood of 0. Define $\theta_x(x) = \theta(x/x)$. Prove that if $\mu \in \mathcal{D}'(\mathbb{R})$ then $\theta_x \mu \to \mu$ in $\mathcal{D}'(\mathbb{R})$. 
Example
A function \( f : \mathbb{R} \to \mathbb{C} \) is \underline{refinable} if it satisfies a refinement equation of the form

\[
f(x) = \sum_{k \in \mathbb{Z}} c_k f(2x - k)
\]

for some scalars \( c_k \) (the dilation factor 2 is chosen for convenience only, other dilations may be used). See the figures for some examples of refinable functions.

A refinable function is a fixed point of the operator

\[
Sf(x) = \sum_{k \in \mathbb{Z}} c_k f(2x - k).
\]

Often, an iteration \( f_{n+1}(x) = S f_n(x) \) will converge to a solution \( f \) (if one exists). This exercise, illustration where things go wrong - this Cascade Algorithm does not converge in a functional sense. On the other hand, it does converge distributionally.

a. Show that \( f = X_{[0,3]} \) is a solution to the refinement equation

\[
f(x) = f(2x) + f(2x - 3).
\]

b. Let \( f_0 = X_{[0,1]} \). Compute \( f_1, f_2, f_3 \) using the Cascade algorithm.
c. Show that

\[ f_n = \sum_{k=0}^{2^n - 1} \chi_{\left[ \frac{3k}{2^n}, \frac{3k+1}{2^n} \right]} \]

d. Prove that \( f_n \to \frac{1}{3} \chi_{[0,3]} \) in \( \mathcal{D}'(\mathbb{R}) \), i.e.,

\[ \forall h \in C_c^\infty(\mathbb{R}), \quad \langle h, f_n \rangle \to \frac{1}{3} \int_0^3 h(x) \, dx. \]
Motivation

To motivate our next result, suppose that $f, g \in C^\infty(\mathbb{R})$, and that $\mu$ is a function, say $\mu \in L'(\mathbb{R})$. Then

$$\langle f, g \ast \mu \rangle = \int f(x) \overline{(g \ast \mu)(x)} \, dx$$

$$= \int f(x) \overline{\int g(x-y) \mu(y) \, dy} \, dx$$

$$= \int \left( \int f(x) \overline{\tilde{g}(y-x)} \, dx \right) \mu(y) \, dy$$

$$= \int \overline{(f \ast \tilde{g})(y)} \mu(y) \, dy$$

$$= \langle f \ast \tilde{g}, \mu \rangle,$$

where $\tilde{g}(x) = \overline{g(-x)}$. The interchange of integrals is justified by Fubini's Theorem.

Our next result shows that this equality extends to arbitrary distribution $\mu \in D'(\mathbb{R})$.

Thus, this can be viewed as an equivalent definition of the convolution of $g \in C^\infty(\mathbb{R})$ with $\mu \in D'(\mathbb{R})$. 

Theorem

If \( \mu \in \mathcal{D}'(\mathbb{R}) \), then

\[
\forall f, g \in C^\infty(\mathbb{R}), \quad \langle f, g \ast \mu \rangle = \langle f \ast \tilde{g}, \mu \rangle.
\]

Remark: By an earlier exercise, \( g \ast \mu \in C^\infty(\mathbb{R}) \), so the inner product \( \langle f, g \ast \mu \rangle \) is well-defined since \( f \in C^\infty(\mathbb{R}) \). Likewise, \( f \ast \tilde{g} \in C^\infty(\mathbb{R}) \), so \( \langle f \ast \tilde{g}, \mu \rangle \) is defined.

Proof:

Let \( T \) be s.t. \( \text{supp}(f) \subseteq [-T, T] \). Let

\[
X_0 = -T < x_1 < \cdots < x_N = T \quad \text{be a regular partition of } [-T, T], \quad \text{and set } \Delta x = \frac{2T}{N}.
\]

Then, since \( f \in C^\infty(\mathbb{R}) \) and \( g \ast \mu \in C^\infty(\mathbb{R}) \), we can write \( \langle f, g \ast \mu \rangle \) as a limit of Riemann sums:

\[
\langle f, g \ast \mu \rangle = \int_{-T}^{T} f(x) \langle T_x \tilde{g}, \mu \rangle \, dx
\]

\[
= \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k) \langle T_{x_k} \tilde{g}, \mu \rangle \Delta x
\]

\[
= \lim_{N \to \infty} \langle \sum_{k=1}^{N} f(x_k) T_{x_k} \tilde{g} \Delta x, \mu \rangle
\]

\[
= \lim_{N \to \infty} \langle h_N, \mu \rangle
\]
We claim that \( h_N = \sum_{k=1}^{N} f(x_k) T_{x_k} \tilde{g} \cdot \Delta x \rightarrow f*\tilde{g} \)

in \( C^\infty_c(M) \) as \( N \rightarrow \infty \). First, since \( x_k \in [-T, T] \), we have

\[ \text{supp}(h_N) \subseteq \text{supp}(\tilde{g}) + [-T, T] = K, \]

which is a compact set independent of \( N \).

Second, we can also write \( f*\tilde{g} \) in terms of Riemann sums:

\[ f*\tilde{g}(t) = \int_{-T}^{T} f(x) \tilde{g}(t-x) \, dx = \int_{-T}^{T} f(x) T_{x} \tilde{g}(t) \, dt = \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k) T_{x_k} \tilde{g}(t) \, \Delta x. \]

In fact (exercise), because \( f, g \in C^\infty_c(M) \), this convergence is uniform, i.e.,

\[ \| f*\tilde{g} - h_N \|_\infty = \| f*\tilde{g} - \sum_{k=1}^{N} f(x_k) T_{x_k} \tilde{g} \cdot \Delta x \|_\infty \rightarrow 0 \text{ as } N \rightarrow \infty. \]

And since differentiation commutes with convolution, by the same argument we have for any \( n \geq 0 \),

\[ (\partial^n f)*\tilde{g} \rightarrow (f*\tilde{g})_n \text{ as } N \rightarrow \infty. \]
\[ \| (f * \tilde{g})^{(m)} - h_{N}^{(m)} \|_{\infty} \]

\[ = \| f * \tilde{g}^{(m)} - \sum_{k=1}^{N} f(x_k) T_{x_k} \tilde{g}^{(m)} \|_{\infty} \Delta x \|_{\infty} \]

\[ \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty \]

Thus, \( h_N \rightarrow f * \tilde{g} \) in \( C^\infty(M) \). But \( \mu \) is continuous, so this implies that

\[ \langle f, g*\mu \rangle = \lim_{N \rightarrow \infty} \langle h_N, \mu \rangle = \langle f * \tilde{g}, \mu \rangle. \]

We can use this theorem to obtain some interesting results. First, despite the fact that \( C_c^\infty(M) \) appears to be a very small part of \( \mathcal{D}'(M) \), we show that it is in fact dense in \( \mathcal{D}'(M) \) in the topology of \( \mathcal{D}'(M) \). That is, every distribution is a weak* limit of functions in \( C_c^\infty(M) \).
Theorem
\( C^\infty_c(\mathbb{M}) \) is dense in \( \mathcal{D}'(\mathbb{R}) \) (in \( \mathcal{D}'(\mathbb{R}) \) topology).

That is,
\[ \forall \mu \in \mathcal{D}'(\mathbb{R}), \quad \exists \text{ net } \{f_i\} \subseteq C^\infty_c(\mathbb{M}) \text{ s.t. } \]
\[ \forall \varphi \in C^\infty_c(\mathbb{R}), \quad \langle \varphi, f_i \rangle \to \langle \varphi, \mu \rangle. \]

Proof:
Let \( \Theta_n \in C^\infty_c(\mathbb{M}) \) be a smooth cutoff function:

\[ \text{where } \supp(\Theta_n) \subseteq [-n-1, n+1], \quad 0 \leq \Theta_n \leq 1, \quad \Theta_n = 1 \text{ on } [-n, n]. \]

Then for given \( \mu \in \mathcal{D}'(\mathbb{M}) \), we have for any \( \varphi \in C^\infty_c(\mathbb{M}) \) that \( \forall \) large enough \( n \),
\[ \langle \varphi, \mu \Theta_n \rangle = \langle \varphi \Theta_n, \mu \rangle = \langle \varphi, \mu \rangle. \]

Specifically, \( \Theta_n \) holds \( \forall n \) with \( \supp(\varphi) \subseteq [-n, n] \).

In other words, \( \mu \Theta_n \xrightarrow{w^*} \mu \).

Now choose \( k \in C^\infty_c(\mathbb{M}) \) with \( \int k = 1 \), & set \( k_\lambda(x) = \lambda k(\lambda x) \).

Let \( I = \mathbb{N} \times (0, \infty) \), under \& ordering...
\((n_1, \lambda_1) \leq (n_2, \lambda_2) \iff n_1 \leq n_2 \& \lambda_1 \leq \lambda_2.\)

Then \( \{ k_x \ast \mu \Theta_n \}_{(n, x) \in I} \) is a net in \( C^\infty(M) \).

Choose any \( \Psi \in C^\infty(M) \).

Let \( n_0 \) be s.t. \( \text{supp}(\Psi) \subseteq [-n_0, n_0] \). Since \( \{ \tilde{k}_x \}_{x > 0} \) is an approximate identity (as is \( \{ k_x \}_{x > 0} \)), choose \( \varepsilon > 0 \).

\( \Psi \ast \tilde{k}_x \to \Psi \text{ in } C^\infty(M). \) As \( \mu \) is continuous, this implies \( \langle \Psi \ast \tilde{k}_x, \mu \rangle \to \langle \Psi, \mu \rangle \), so

\[ \exists \lambda_0 > 1 \text{ s.t.} \]

\[ \lambda > \lambda_0 \implies | \langle \Psi, \mu \rangle - \langle \Psi \ast \tilde{k}_x, \mu \rangle | < \varepsilon. \]

Note that if \( \lambda > \lambda_0 > 1 \), then \( \text{supp}(\Psi \ast \tilde{k}_x) \subseteq [-n_0-1, n_0+1] \).

Hence if \((n_1, \lambda_1) \succ (n_0+1, \lambda_0)\), then

\[ | \langle \Psi, \mu \rangle - \langle \Psi, k_x \ast \mu \Theta_n \rangle | \]

\[ = | \langle \Psi, \mu \rangle - \langle \Psi \ast \tilde{k}_x, \mu \Theta_n \rangle | \] (preceding theorem)
\[ \left| \langle \psi, \mu \rangle - \langle (\psi * \bar{k}_2) \Theta_n, \mu \rangle \right| = \left| \langle \psi, \mu \rangle - \langle \psi * \bar{k}_2, \mu \rangle \right| \]

< \varepsilon.

Thus \( k_n * \mu \Theta_n \overset{\text{weak*}}{\to} \mu \).

Remark

Weak* convergence is a "very weak" type of convergence, & thus the density of \( \mathcal{C}_0^\infty(\mathbb{R}) \) in \( \mathcal{D}'(\mathbb{R}) \) is only a very weak statement.

However, our next result is very practical: it shows that classical & distributional differentiation of functions coincide for all continuously differentiable functions.
An earlier exercise shows that convolution commutes with translations. We now expand on that result.

**Theorem** (see Rudin, F. A., 2nd ed., p. 173)

a. If $\mu \in \mathcal{O}'(\mathbb{R})$, then

$$L : C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$$

$$f \mapsto f \star \mu$$

is a continuous linear map of $C_c^\infty(\mathbb{R})$ into $C_c^\infty(\mathbb{R})$ that satisfies $LT_a = T_a L$ for all $a \in \mathbb{R}$.

b. If $L : C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$ is a continuous map s.t. $LT_a = T_a L \forall a \in \mathbb{R}$, then there is a unique $\mu \in \mathcal{O}'(\mathbb{R})$ s.t. $Lf = f \star \mu$.

**Proof**

a. We have seen before that $L$ is well-defined & we have

$$T_a (Lf) = T_a (f \star \mu) = T_a f \star \mu = L (T_a f).$$

So, it remains only to show that $L$ is continuous.

Suppose that $f_k \to 0$ in $C_c^\infty(\mathbb{R})$, i.e., $f_k$ compact $K \in \mathbb{R}$ s.t. $\text{supp}(f_k) \subseteq K$ $\forall k$, and
\[ \forall n > 0, \quad \| f_k^{(n)} \|_\infty \to 0. \]

We must show that \( f_k \ast \mu \to 0 \) in \( C^\infty(\mathbb{R}) \), i.e., for any compact \( Q \subseteq \mathbb{R} \) we must show that

\[ \forall n > 0, \quad \| (f_k \ast \mu)^{(n)} \|_Q \to 0. \]

Now, \( (f_k \ast \mu)(x) = \left\langle T_x f_k, \mu \right\rangle \), and
\[ \text{supp}(T_x f_k) = x - K. \]

The set
\[ Q - K = \{ x - y : x \in Q, y \in K \} \]

is compact, so since \( \mu \) is continuous, by an earlier theorem \( \exists N > 0, \exists C > 0 \) s.t.
\[ f \in C^\infty(Q - K) \Rightarrow \left| \left\langle f, \mu \right\rangle \right| \leq C \sum_{n=0}^{\infty} \| f^{(n)} \|_\infty. \]

Hence if \( x \in Q \) then since \( \text{supp}(T_x f_k) \subseteq x - K \subseteq Q - K \),
\[ |(f_k \ast \mu)(x)| = \left| (f_k^{(m)} \ast \mu)(x) \right| \\
= \left| \left\langle T_x f_k^{(m)}, \mu \right\rangle \right| \]
\[
\leq C \sum_{n=0}^{N} \| (T_{x}f_{k}^{(m^n)})^{(n)} \|_{\infty}
\]
\[
= C \sum_{n=0}^{\infty} \| T_{x}f_{k}^{(m^n)} \|_{\infty}
\]
\[
= C \sum_{n=0}^{\infty} \| f_{k}^{(m^n)} \|_{\infty}
\]

Therefore
\[
\sup_{x \in Q} |(f_{k} * x^{(m^n)})(x)| \leq C \sum_{n=0}^{N} \| f_{k}^{(m^n)} \|_{\infty}
\]
\[
\rightarrow 0 \text{ as } k \rightarrow \infty.
\]

Thus \(L\) is continuous.

b. Suppose \(L: C^{\infty}(\mathbb{R}) \rightarrow C^{\infty}(\mathbb{R})\) is linear & continuous &
commutes with translations. Define \(\mu: C^{\infty}(\mathbb{R}) \rightarrow \mathbb{C}\) by
\[
\langle f, \mu \rangle = \frac{L \hat{f}(0)}{2 \pi}, \quad f \in C^{\infty}(\mathbb{R}).
\]
Then \(\mu\) is a linear functional on \(C^{\infty}(\mathbb{R})\) & \(\hat{f}_{k} \rightarrow 0\)
in \(C^{\infty}(\mathbb{R})\) then \(L \hat{f}_{k} \rightarrow 0\) in \(C^{\infty}(\mathbb{R})\) since \(L\) is continuous.

In particular, \(\hat{f}_{k}\) convergence is uniform so
\[
\langle f_{k}, \mu \rangle = \frac{L \hat{f}_{k}(0)}{2 \pi} \rightarrow 0.
\]
Thus \(\mu \in \mathcal{D}'(\mathbb{R})\). Further
\((f \ast \mu)(x) = \langle T_x \tilde{f}, \mu \rangle \)

\[= L \langle (T_x \tilde{f})^\sim \rangle (0) \]

\[= L (T_{-x} f)(0) \]

\[= T_{-x} (Lf)(0) \]

\[= Lf(x). \]

Exercise: Show \( \mu \) is unique.
Corollary (see Rudin, F.A., 2nd ed., p. 180)

Suppose \( L : C_c^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}) \) is a continuous linear map that commutes with differentiation, i.e., \( LD = DL \). Then \( \exists \mu \in \mathcal{D}'(\mathbb{R}) \) s.t.

\[ Lf = f * \mu \text{ for all } f \in C_c^\infty(\mathbb{R}). \]

Proof:

Choose any \( f \in C_c^\infty(\mathbb{R}) \). Define

\[ g(x) = T_{-x} LT_x f (0) = LT_x f (x). \]

By a previous exercise, we know that \( \frac{T_{-x}f}{-h} \rightarrow f' \)

in \( C_c^\infty(\mathbb{R}) \) as \( h \rightarrow 0 \). Hence \( \frac{T_{-x}f}{-h} \rightarrow T{xf'} \) in \( C^\infty(\mathbb{R}) \)

so, since \( L \) is linear & continuous,

\[ \frac{LT_{x+h}f - LT_x f}{-h} \rightarrow LT_x f' \text{ in } C^\infty(\mathbb{R}). \]

In particular, \( \& \text{ LHS is converging uniformly to the RHS,} \)

which implies
\[
\frac{L T_{x+h} f(x+h) - L T_x f(x+h)}{-h} \to L T_x f'(x)
\]

Therefore, letting \( h \to 0 \),

\[
g'(x) = \frac{L T_{x+h} f(x+h) - L T_x f(x)}{h}
\]

\[
= \frac{L T_{x+h} f(x+h) - L T_x f(x+h)}{h} + \frac{L T_x f(x+h) - L T_x f(x)}{h}
\]

\[
\to -L T_x f'(x) + D L T_x f(x)
\]

\[
= -L D T_x f(x) + D L T_x f(x)
\]

\[
= 0
\]

Hence \( g'(x) = 0 \ \forall x \), so \( g \) is constant. Therefore

\[
T_x L T_x f(x) = g(x) = g(0) = L f(0).
\]

Since this is true for every \( f \in C^0(\mathbb{R}) \), we can replace \( f \) by \( Tyf \) to get

\[
T_x L T_x f(y) = L f(y) \ \forall y.
\]

Hence \( T_x L T_x = L \), & therefore the result follows from the preceding theorem.