3.5 "Generalized function" properties: Differentiation

Many properties valid for functions in $C^\infty(M)$ can be extended to its dual space $\mathcal{D}'(M) = C_c^\infty(M)^\ast$. For example, we can define the derivative of any distribution.

Motivation:
Suppose that $g \in C^0(M) = L^1_{\text{loc}}(M) \subseteq \mathcal{D}'(M)$. Then $g$ determines a distribution $T_g$. But furthermore, $g'$ is a continuous function, so $g' \in C(M) \subseteq L^1_{\text{loc}}(M) \subseteq \mathcal{D}'(M)$. Thus $g'$ determines a distribution $T_{g'}$, and by integration by parts we have for $f \in C_c^\infty(M)$ that

\[
\langle f, T_{g'} \rangle = \int_{-\infty}^{\infty} f(x) g'(x) \, dx
\]

\[
= f(x)g(x) \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) g(x) \, dx
\]

\[
= 0 - \langle f', T_g \rangle.
\]

We define the distribution $(T_g)'$ to be given by

\[
\langle f, (T_g)' \rangle \overset{df}{=} -\langle f', T_g \rangle = -\langle f, T_g \rangle.
\]

That is, we define $T_g' = -T_g$. 
**Definition**

The derivative of $T \in \mathcal{D}'(\mathbb{R})$ is $T' : \mathcal{C}_c(\mathbb{R}) \to \mathbb{C}$ given by

\[ \langle f, T' \rangle = -\langle f', T \rangle, \quad f \in \mathcal{C}_c(\mathbb{R}). \]

In essence, we declare that integration by parts is valid for distributions.

**Exercise**

a. Show that if $T \in \mathcal{D}'(\mathbb{R})$ then $T' \in \mathcal{D}'(\mathbb{R})$.

b. Show that if $g \in \mathcal{C}^1(\mathbb{R})$ then ordinary and distributional differentiation are equal, i.e.,

\[ (Tg)' = Tg'. \]

c. Show that $\langle f, \delta' \rangle = -f'(0)$, $\text{supp}(\delta') = \{0\}$.

d. Show that $\delta$ is infinitely differentiable in the sense of distributions, and find a formula for $\delta^{(n)}$.

e. Show that every $T \in \mathcal{D}'(\mathbb{R})$ is infinitely differentiable in the sense of distributions.

f. Let $H$ be the Heaviside function: $H = \chi_{[0,\infty)} \in \mathcal{D}'(\mathbb{R})$.

Show that the distributional derivative of $H$ is $H' = \delta$, but the pointwise a.e. derivative is $H'\neq 0$ a.e.
Notation
To distinguish between the ordinary and distributional derivatives of functions, we will write

\[ Dg = \text{distributional derivative of } g \in L^1(\mathbb{R}) \]

\[ g' = \text{pointwise a.e. derivative of a function that is differentiable a.e.} \]

Theorem (see Benedetto, p. 81)
Assume \( g \) is continuously differentiable on \( \mathbb{R} \setminus \{0\} \), & has a jump discontinuity at 0 of height \( \sigma = g(0^+) - g(0^-) \). Then, as distributions,

\[
Dg = g' + \sigma \delta
\]

That is,

\[
\forall f \in C_c, \quad \langle f, Dg \rangle = \int \left( f(x) g'(x) - \sigma f(x) \right) dx
\]

Proof: Exercise.

Hints: Write

\[
\langle f, Dg \rangle = - \langle f', g \rangle = - \int_{-\infty}^{\infty} f(x) g'(x) dx
\]

\[ - \int_{-\infty}^{\infty} f'(x) g(x) dx \]
(More precisely, these are improper Riemann integrals.)

Since $f'$ & $g$ are continuously differentiable on $\mathbb{R} \setminus \{0\}$, you can apply integration by parts. Recall that $f$ is compactly supported, hence $fg$ is as well.
Exercise (see Benedetto, p. 82)

Define

\[ g(x) = \begin{cases} \frac{1}{x} e^{-x^2}, & x > 0 \\ 0, & x \leq 0 \end{cases} \]

Show that \( g \) is continuous on \( \mathbb{R} \), \( g \) is continuously differentiable on \( \mathbb{R} \setminus \{0\} \), but at points where \( g \) does not belong to \( L^1_{\text{loc}}(\mathbb{R}) \). How does this relate to the preceding exercise?
Exercise
Let \( g(x) = \ln|x| \) (defined everywhere except \( x = 0 \)).
Show that \( \ln|x| \in L^1_{\text{loc}}(\mathbb{R}) \).

Hints: Given \( 0 < a < 1 < b < \infty \), write
\[
\int_a^b |\ln x| \, dx = -\int_a^1 \ln x \, dx + \int_1^b \ln x \, dx.
\]
Integrate using the fact that \( \frac{d}{dx} (x \ln x - x) = \ln x \).
Taking the limit as \( a \to 0 \), conclude that
\[
\int_0^b |\ln x| \, dx < \infty \quad \forall b > 1.
\]

Example
Since \( \ln|x| \in L^1_{\text{loc}}(\mathbb{R}) \), it has a distributional derivative. We will show that
\[
D(\ln|x|) = \text{pv}(\frac{1}{x}).
\]

By definition, if \( f \in C^\infty(\mathbb{R}) \), then
\[
\langle f, D(\ln|x|) \rangle = -\langle f', \ln|x| \rangle = -\int f'(x) \ln|x| \, dx.
\]
Let \( R \) be s.t. \( \text{supp}(f) \subseteq [-R, R] \).
If \( 0 < a < 1 < R < b \), then
\[
\int_a^b f'(x) \ln x \, dx = f(b) \ln b - f(a) \ln a - \int_a^b \frac{f(x)}{x} \, dx
\]
and similarly
\[
\int_{-a}^{-b} f'(x) \ln(x) \, dx = f(-b) \ln b - f(-a) \ln a - \int_{-b}^{-a} \frac{f(x)}{x} \, dx
\]

Hence
\[
\int_{a < |x| < b} f'(x) \ln|x| \, dx = -\ln a \left[ \frac{f(a) - f(0)}{a - 0} + \frac{f(-a) - f(0)}{-a - 0} \right]
\]

Thus
\[
\lim_{a \to 0^+, b \to \infty} \int_{a < |x| < b} f'(x) \ln|x| \, dx = \left( \lim_{a \to 0^+} \ln a \right) \left[ \lim_{a \to 0^+} \frac{f(a) - f(0)}{a - 0} \right] + \left( \lim_{a \to 0^+} \frac{f(-a) - f(0)}{-a - 0} \right)
\]

\[
\lim_{a \to 0^+, b \to \infty} \int_{a < |x| < b} f(x) \, dx
\]
\[ = 0 \cdot (f'(0) + f''(0)) + \langle f, p(u(x)) \rangle \]

Thus
\[ \langle f, D(ln|x|) \rangle = \langle f, p(u(x)) \rangle \forall f \in C^\infty(R). \]

\[ \square \]
Exercises

Compute & distributional derivatives of the following functions.

a. $\mathcal{H}_{[-1,1]}$

b. $H(x) \cos x$, where $H = \mathcal{H}_{[0,\infty)}$ is the Heaviside function.

c. $xH(x)$.

d. $|x|$

e. $|\cos x|$

Exercise

Define $x^n \delta^{(m)}(x)$ in the usual way, i.e.,

$$\langle f, x^n \delta^{(m)}(x) \rangle = \langle x^m f(1/x), \delta^{(m)} \rangle, \quad f \in C^\infty(\mathbb{R})$$

Prove that $x^n \delta^{(m)}(x) = (-1)^n n! \delta$.

Exercise

Show $\exists g \in C^\infty(\mathbb{R})$ s.t. $D^2 g = \delta$. 
Exercise (see Rudin, F.A., 2nd ed., p. 178)

a. Let \( \mu \in \mathcal{D}'(\mathbb{R}) \) & \( \theta \in C^\infty(\mathbb{R}) \) be given. Show
\[
\theta = 0 \text{ on } \text{supp}(\mu) \nRightarrow \theta \mu = 0.
\]
Hint: Consider \( \theta \xi' \).

b. Let \( \mu \in \mathcal{D}'(\mathbb{R}) \) be s.t. \( \text{supp}(\mu) \subseteq [-1,1] \). Show
\[
\theta = 0 \text{ on } [-1,1] \nRightarrow \theta \mu = 0.
\]
Explain the difference from part a.
Exercise
Let $\mu \in \mathcal{D}'(\mathbb{R})$ & $f \in C^\infty(\mathbb{R})$ be given. Recall that

$$(f \ast \mu)(x) = \langle T_x f, \mu \rangle, \quad \tilde{f}(x) = f(-x).$$

An earlier exercise shows $f \ast \mu \in C^\infty(\mathbb{R})$ & $(f \ast \mu)' = f' \ast \mu$.

Prove that

$$f \ast D\mu = f' \ast \mu = (f \ast \mu)' = D(f \ast \mu).$$
Exercise: Product Rule

Let \( \mu \in \mathcal{D}'(\mathbb{R}) \) & \( \theta \in C^\infty(\mathbb{R}) \). Recall that

\( \theta \mu \) is a distribution defined by \( \langle f, \theta \mu \rangle = \langle f \theta, \mu \rangle \)

for \( f \in C^\infty(\mathbb{R}) \). Prove that

\[ D(\theta \mu) = \theta' \mu + \theta D \mu. \]

Exercise (see Rudin, FA, 2nd ed., p. 186)

Given \( \mu \in \mathcal{D}'(\mathbb{R}) \), prove that

\[ \frac{\mu - \theta \mu}{a} \to D \mu \quad \text{in} \quad \mathcal{D}'(\mathbb{R}) \quad \text{as} \quad a \to 0. \]
Lemma
Suppose that \( g \in C^1(\mathbb{R}) \) is such that \( Dg \in C(\mathbb{R}) \). Then \( g \in C^1(\mathbb{R}) \) and \( g' = Dg \).

Remark
Since \( C(\mathbb{R}) \subseteq L^1_{\text{loc}}(\mathbb{R}) \subseteq D'(\mathbb{R}) \), we know that \( g \) has a distributional derivative. By saying that \( Dg \in C(\mathbb{R}) \), we are saying that \( f \) is a continuous function \( G \) that acts as a distribution just as \( Dg \) does, i.e.,

\[
\forall f \in C_c^\infty(\mathbb{R}), \quad \langle f, Dg \rangle \overset{\text{def}}{=} -\langle f', g \rangle = \langle f, G \rangle
\]

Proof:
Note that if \( g \) did belong to \( C^1(\mathbb{R}) \), then by the Fundamental Theorem of Calculus, we would have

\[
g(x+h) - g(x) = \int_x^{x+h} g'(t) \, dt
\]

\[
= \int_0^h g'(x+t) \, dt
\]

\[
= h \int_0^1 g'(x+ht) \, dt.
\]
We do not yet know that $g'$ exists, but we claim that:

\[(*) \quad g(x+h) - g(x) = \int_x^{x+h} Dg(t) \, dt = h \int_0^1 Dg(x+ht) \, dt.\]

Before proving our claim, observe that if $f \in C_c^\infty(\mathbb{R})$,

then $\tilde{f} * g \in C_c^\infty(\mathbb{R})$, & by the FTC we have

\[h \int_0^1 (\tilde{f} * g)'(ht) \, dt = \int_0^1 (\tilde{f} * g)'(t) \, dt\]

\[= (\tilde{f} * g)(h) - (\tilde{f} * g)(0)\]

\[= (\tilde{f} * g)(h) - \langle f, g \rangle.\]

By an earlier exercise, $(\tilde{f} * g)' = \tilde{f} * Dg$, so we have

\[\langle f, g \rangle = (\tilde{f} * g)(h) - h \int_0^1 (\tilde{f} * Dg)(ht) \, dt\]

\[= \int f(-x) \overline{g(h-x)} \, dx - h \int_0^1 f(-x) \overline{Dg(ht-x)} \, dx \, dt\]

\[= \int f(x) \overline{g(h+x)} \, dx - h \int_0^1 f(x) \overline{Dg(ht+x)} \, dx \, dt.\]

Exercise: Apply Fubini's Theorem to justify the interchange in the order of integration.

Since $\langle f, g \rangle = \int \overline{f(x)} g(x) \, dx$, rearranging the above calculation gives
\[
\int f(x) \left[ g(x+h) - g(x) - h \int_0^1 Dg(x+ht) \, dt \right] \, dx = 0.
\]

Since the term in brackets is a continuous function of \(x\), and since (***) holds for every \(f \in C^\infty_c(\mathbb{R})\), the term in brackets must be identically zero. This proves that the claimed equation (**) holds, and hence

\[
\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} Dg(t) \, dt. \quad (***)
\]

Since \(Dg\) is continuous, it follows from the Fundamental Theorem of Calculus that the RHS of (***), as \(h \to 0\), approaches \(Dg(x)\). Thus

\[
g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} Dg(t) \, dt = Dg(x).
\]
Exercise

In the course of the preceding proof, we showed that if \( g \in C_c(\mathbb{R}) \) and \( f \in C^\infty(\mathbb{R}) \), then

\[
\langle f, g \rangle = \left( \hat{f} * g \right)(h) - h \left. \int_0^1 \left( \hat{f} * Dg \right)(ht) \, dt \right. .
\]

Prove that this extends to arbitrary distributions, i.e., show that if \( \mu \in \mathcal{D}'(\mathbb{R}) \) and \( h \in \mathbb{R} \) then

\[
\langle f, \mu \rangle = \left( \hat{f} * \mu \right)(h) - h \int_0^1 \left( \hat{f} * D\mu \right)(ht) \, dt.
\]

Hint: By an earlier exercise, \( \hat{f} * D\mu = \left( \hat{f} * \mu \right)' \), and \( \hat{f} * \mu \in C^\infty(\mathbb{R}) \) so the FTC is valid.
Theorem

Let \( \mu \in D'(\mathbb{R}) \) be given. Then

\[ \mu \in C^1(\mathbb{R}) \iff D\mu \in C(\mathbb{R}). \]

Moreover, in this case we have \( D\mu = \mu' \).

Proof:

\[ \Rightarrow \] Suppose \( \mu \) is continuously differentiable on \( \mathbb{R} \), and fix any \( \psi \in C_0^\infty(\mathbb{R}) \). Then integration by parts is valid & gives (say supp(\( \psi \)) \( \subseteq [-T,T] \))

\[ \langle f, D\mu \rangle = - \langle f', \mu \rangle \]

\[ = - \int_{-T}^{T} f'(x) \mu(x) \, dx \]

\[ = -f(x)\mu(x) \bigg|_{-T}^{T} + \int_{-T}^{T} f(x)\mu'(x) \, dx \]

\[ = 0 + \langle f', \mu' \rangle. \]

Hence \( D\mu = \mu' \in C(\mathbb{R}) \) (equality of distributions).
Suppose that \( \mu \in L'(\mathbb{R}) \) and \( D\mu \in C(\mathbb{R}) \).

Our first goal is to show that \( \mu \in C(\mathbb{R}) \).

Once this is done, an earlier lemma implies that \( \mu \in C^1(\mathbb{R}) \) and \( D\mu = \mu' \), so the proof is complete.

From an earlier exercise, we have for any \( f \in C_c^\infty(\mathbb{R}) \) and \( \forall x \in \mathbb{R} \),

\[
\langle f, \mu \rangle = \langle \tilde{f} \ast \mu, x \rangle = \int_0^\infty x (\tilde{f} \ast \mu)(xy) \, dy,
\]

and also \( (\tilde{f} \ast \mu)' = \tilde{f} \ast D\mu \) by an earlier exercise.

Choose any function \( k \in C_c^\infty(\mathbb{R}) \) with \( \int k = 1 \).

Then

\[
\langle f, \mu \rangle = \int \langle f, \mu \rangle \, k(x) \, dx
\]

\[
= \int \int k(x)(\tilde{f} \ast \mu)(x) \, dx \, dy - \int k(x) \int_0^\infty (\tilde{f} \ast D\mu)(xy) \, dy \, dx
\]

\[
= \langle k, \tilde{f} \ast \mu \rangle - \int \int k(x) \tilde{t}(x) D\mu(xy-t) \, dt \, dy \, dx
\]
\[ = \langle k \ast f, \mu \rangle - \int f(-t) \int_0^1 x^2 k(x) D\mu(xy-t) \, dx 
\]
\[ = \langle k \ast f, \tilde{\mu} \rangle - \int f(t) \int_0^1 x^2 k(x) D\mu(xy+t) \, dx 
\]
\[ = \int f(t) \left[ \tilde{\mu}(t) - \int_0^1 x^2 k(x) D\mu(xy+t) \, dx \right] \, dt 
\]

Hence, \[ \psi \] as distributions, \( \mu \) equals \( \tilde{\mu} \) function

\[ g(t) = (k \ast \mu)(t) - \int_0^1 \bar{x} k(x) D\mu(xy+t) \, dx 
\]

The first term on the right belongs to \( C^\infty(\mathbb{R}) \), &

the second is a continuous function of \( t \), so

\[ \mu = g \in C(\mathbb{R}) \] & the proof is complete. 

Exercise: Use Fubini's Theorem to justify & interchange of integrals in the proof.

\[ \text{Exercise: Show that if } \mu \in D'(\mathbb{R}) \text{ & } D\mu = 0, \]

then \( \mu \) equals a constant function.
Motivation
If \( g \in C(\mathbb{R}) \) then \( g \in L^1_{\text{loc}}(\mathbb{R}) \), hence determines a distribution, and has distributional derivatives \( \frac{\partial^k g}{\partial x^k} \) of every order \( k > 0 \), even if \( g \) is not differentiable.

The next result is a partial converse to this statement. Namely, every distribution is "locally" a derivative of some continuous function.

In order to prove this, we recall a result proved in the appendix giving an analogue of "continuity = boundedness" for distributions. Specifically, it is shown in the Appendix that a linear map
\[
\mu : C^\infty(\mathbb{R}) \to \mathbb{C}
\]
is continuous if and only if
\[
\forall \text{ compact } K \subseteq \mathbb{R}, \exists N_k > 0, \exists C_k > 0 \text{ s.t. } \quad f \in C^\infty(\mathbb{R}), \supp(f) \subseteq K \Rightarrow |\langle f, \mu \rangle| \leq C_k \|f\|_{N_k},
\]
where
\[ \|f\|_N = \max \{\|f\|_\infty, \|f'\|_\infty, \ldots, \|f^{(n)}\|_\infty \} \]

If a single \( N \) will work for every compact set \( K \) (with different \( C_k \)) then the smallest such \( N \) is called the order of \( \mu \).
Theorem
Let \( T \in \mathcal{D}'(\mathbb{R}) \) & a compact \( K \subseteq \mathbb{R} \) be given.

Then \( \exists g \in C_c(\mathbb{R}) \), \( \exists k \geq 0 \) s.t.

\[ \forall f \in C_c^\infty(\mathbb{R}) \text{ with } \text{supp}(f) \subseteq K, \quad \langle f, T \rangle = \langle f, D^k g \rangle. \]

Remark
\[ \langle f, D^k g \rangle = (-1)^k \langle f^{(k)}, g \rangle = (-1)^k \int f^{(k)}(x) g(x) \, dx. \]

Proof:
By dilating \( T \) we may assume \( K \subseteq [0, 1] \) (exercise).

Then \( \exists N \geq 0, \exists C > 0 \) s.t. \( \forall f \in C_c^\infty(\mathbb{R}) \text{ with } \text{supp}(f) \subseteq K, \)

\[ |T(f)| \leq C \| f \|_N. \]

For such \( f \), we have by the Mean Value Theorem that if \( 0 \leq x \leq 1 \) \& \( \exists c \in (0, x) \) s.t.

\[ f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} \quad (\text{since } \text{supp}(f) \subseteq [0, 1]) \]

Hence

\[ |f(x)| = |x| |f'(c)| \leq |f'(c)| \leq \| f' \|_\infty. \]

This is also valid for \( x \notin [0, 1] \) since \( f(x) = 0 \) in that case.

Thus

\[ \| f \|_\infty \leq \| f' \|_\infty. \]
We can repeat the argument to obtain

$$\|f\|_\infty \leq \|f'\|_\infty \leq \cdots \leq \|f^{(n)}\|_\infty,$$

and hence

$$\|f^{(n)}\|_\infty = \max \{\|f\|_\infty, \ldots, \|f^{(n)}\|_\infty\} = \|f^{(n)}\|_\infty.$$

Further, by the Fundamental Theorem of Calculus,

$$f^{(n)}(x) = \int_0^x f^{(n+1)}(t) \, dt,$$

so

$$\|f^{(n)}\|_\infty \leq \sup_{0 \leq x \leq 1} \int_0^x |f^{(n+1)}(t)| \, dt = \|f^{(n+1)}\|_1.$$

Define

$$C_0^\infty(K) = \{f \in C_0^\infty(\mathbb{R}) : \text{supp}(f) \subseteq K\}$$

and

$$L_1^1(K) = \{f \in L_1^1(\mathbb{R}) : \text{supp}(f) \subseteq K\}.$$

Then $D^{n+1} : C_0^\infty(K) \to C_0^\infty(K)$ is linear and injective (exercise). Let

$$Y = \text{range}(D^{n+1}) \subseteq C_0^\infty(K).$$

Define

$$\Lambda : Y \to C$$

by $D^{n+1}f \mapsto \langle f, T \rangle$. 
Then \[ \left| \frac{\alpha^n}{n!} \right| = \left| \langle f, T \rangle \right| \leq C \| f \|_W \]
\[ \leq C \| f^{(n+1)} \|_1 \]
\[ = C \| D^{n+1} f \|_1. \]

Thus, considering \( Y \) as a subspace of \( L^1_k(\mathbb{R}) \)
under the \( L^1 \)-norm, \( \Lambda : Y \to C \) is continuous.
The Hahn-Banach Theorem therefore implies that \( \Lambda \) extends to a continuous mapping on all of \( L^1_k(\mathbb{R}) \).

Thus, \( \Lambda \in L^1_k(\mathbb{R})^* = L^\infty_k(\mathbb{R}), \) i.e.,
\[ \forall \Psi \in L^\infty(\mathbb{R}) \text{ with } \text{supp}(\Psi) \subseteq K \text{ s.t.} \]
\[ \left| \langle f, \Lambda \rangle \right| = \int f(x) \overline{\Psi(x)} \, dx, \quad f \in L^1_k(\mathbb{R}). \]

Define \( g(x) = \int_0^x \Psi(t) \, dt. \)
Then \( g \in C(\mathbb{R}), \) and for \( f \in C^\infty_c(K) \) we have...
\[ \int D^{N+2} f(x) \overline{g(x)} \, dx = - \int D^{N+1} f(x) \overline{\psi(x)} \, dx \]

\[ = - \langle D^{N+1} f, \lambda \rangle \]

\[ = \langle f, T \rangle \]

Thus

\[ \langle f, T \rangle = \langle D^{N+2} f, g \rangle = \langle f, (-1)^{N+2} D^{N+2} g \rangle \]

Exercise

Show that if \( T \in \mathfrak{D}'(\mathbb{R}) \) has order \( N \), then

\[ \exists g \in C(\mathbb{R}) \text{ s.t. } T = D^{N+2} g. \]

Hint: For each compact \( K \), let \( g_K \) be \( \mathbb{R} \)

function constructed in the preceding theorem. Show that

\[ g_K = g_K' \text{ on } K \cap K'. \]

Remark: It is shown in a previous section that all compactly

supported distributions have finite order.
Exercise (See Rudin, F.A., 2nd ed., p. 178)

Suppose $f_k \in L^1_{loc}(\mathbb{R})$, & $K$ compact $K \subset \mathbb{R}$,

$\| f_k \cdot X_k \|_1 \to 0$ as $k \to \infty$.

Prove that $D^n f_k \to 0$ in $C^0(\mathbb{R})$ as $k \to \infty$ for each $N \geq 0$. 
Recall that $AC_{\text{loc}}(\mathbb{R})$ was defined in Chapter 1.

**Definition**

Let $I$ be an interval in $\mathbb{R}$ (bounded or unbounded, including the possibility $\mathbb{R} = \mathbb{R}$). Then $f$ has bounded variation on $I$ if there exists $M > 0$ such that

\[
\sum_{j=1}^{N} |f(x_{j}) - f(x_{j-1})| \leq M.
\]

We let $BV(I)$ be the collection of all such $f$.

We say that $f \in BV_{\text{loc}}(\mathbb{R})$ if $f \in BV[a,b]$ for all $a < b$.

**Exercise**

a. Prove that if $g \in BV(\mathbb{R}) \cap AC_{\text{loc}}(\mathbb{R})$, then $Dg = g'$ in the sense of distributions.

b. Prove that

\[
L'(\mathbb{R}) = \{Dg : g \in BV(\mathbb{R}) \cap AC_{\text{loc}}(\mathbb{R})\}.
\]
Our next result will illustrate the role of the order of a distribution. First we need the following lemma, e.g., see Rudin, Functional Analysis, 2nd ed., Lemma 3.9. (also Lieb & Loss)

Lemma
Let $X$ be a vector space & suppose $\mu_1, \ldots, \mu_N : X \rightarrow \mathbb{C}$ are linear. Set

$$K = \bigcap_{k=1}^{N} \ker(\mu_k).$$

Then TFAE.

a. $\mu = \sum_{k=1}^{N} C_k \mu_k$ for some $C_k \in \mathbb{C}$

b. $\exists C > 0 \text{ s.t. } \forall x \in X,$

$$|\langle x, \mu \rangle| \leq C \max_{k=1, \ldots, N} |\langle x, \mu_k \rangle|$$

c. $x \in K \implies \langle x, \mu \rangle = 0.$

Proof:
Exercises: a $\implies$ b & b $\implies$ c.
c ⇒ a. Assume c holds, & define

\[ \pi : X \to \mathbb{C}^N \]
\[ x \mapsto (\langle x, \mu_1 \rangle, \ldots, \langle x, \mu_N \rangle)^T. \]

Then define

\[ T : \pi(x) \to \mathbb{C} \]
\[ \pi(x) \mapsto \langle x, \mu \rangle. \]

Exercise: Show that the hypothesis of statement c implies that \( T \) is well-defined, i.e.,
\[ \pi(x) = \pi(x') \implies \langle x, \mu \rangle = \langle x', \mu \rangle. \]

Exercise: Show \( T \) is linear.

Exercise: Since \( \pi(X) \) is a finite-dimensional subspace of \( \mathbb{C}^N \), show \( \exists \) linear
\[ \tilde{\pi} : \mathbb{C}^N \to \mathbb{C} \text{ s.t. } \tilde{\pi}(\pi(x)) = T. \]

Now, since \( \tilde{T} \) is linear, it is given by a \( 1 \times N \) matrix, i.e., \( \exists C_1, \ldots, C_N \in \mathbb{C} \) s.t.
\[ \langle u, \tilde{T} \rangle = [C_1 \ldots C_N] \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = \sum_{k=1}^N C_k u_k. \]

for \( u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \in \mathbb{C}^N. \)
Hence for $x \in X$, we have

$$\langle x, \mu \rangle = \langle \pi(x), \overline{\frac{1}{T}} \rangle$$

$$= \langle \pi(x), \overline{\frac{1}{T}} \rangle$$

$$= \begin{bmatrix} c_1 & \cdots & c_N \end{bmatrix} \begin{bmatrix} \langle x, \mu_1 \rangle \\ \vdots \\ \langle x, \mu_N \rangle \end{bmatrix}$$

$$= \sum_{k=1}^{N-1} C_k \langle x, \mu_k \rangle$$

$$= \langle x, \sum_{k=1}^{N-1} C_k \mu_k \rangle.$$
We will show that if a distribution is supported on a single point, then it must be a finite linear combination of derivatives of δ's at that point. To prove this we first need the following lemma.

**Lemma**

If f ∈ C^∞(R) satisfies f(0) = f'(0) = ... = f^{(N)}(0) = 0, then ∀ δ > 0 ∃ δ > 0 s.t.

(1) \( |x| < δ, \ n = 0, ..., N \Rightarrow |f^{(n)}(x)| \leq δ |x|^{N-n} \).

**Proof:**

Since \( f^{(n)} \) is continuous & vanishes at 0, \( \exists δ > 0 \) s.t.

(2) \( |x| < δ, \ n = 0, ..., N \Rightarrow |f^{(n)}(x)| \leq δ \).

We now proceed to prove (1) by induction. If \( n = N \), then (1) is just (2). So, suppose that (1) holds for some \( 0 \leq n < N \). Then given \( |x| < δ \), we have by the Mean Value Theorem that \( f^{(n)}(x) = \frac{f^{(n+1)}(c)}{n!} |x|^n \) for some \( c \) between 0 & \( x \) s.t.
\[
\left| \frac{f^{(n-1)}(x)}{x} \right| = \left| \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x - 0} \right|
\]

\[
= \left| f^{(n)}(0) \right|
\]

\[
\leq \eta \left| x \right|^{N-n}
\]

\[
\leq \eta \left| x \right|^{N-n}.
\]

Hence
\[
\left| f^{(n-1)}(x) \right| \leq \eta \left| x \right|^{N-n+1} = \eta \left| x \right|^{N-(n-1)},
\]

which completes the induction. 

Now we can prove that a distribution supported on a set \( \{a\} \) is a linear combination of derivatives of a \( \delta \) supported at \( a \). By translating, it suffices to consider \( a = 0 \).
Theorem
Given $\mu \in \mathcal{D}'(\mathbb{R})$ TFAE:

a. $\text{supp}(\mu) = \{0\}$.

b. $\exists N \geq 0$, $\exists c_0, \ldots, c_N \in \mathbb{C}$ not all zero s.t.
\[\mu = \sum_{n=0}^{N} c_n D^n f.\]

Proof:
\[b \Rightarrow a.\] Exercise.

\[a \Rightarrow b.\] Suppose $\text{supp}(\mu) = \{0\}$. By a previous theorem, since it has finite order, i.e.,
\[\mu\] has compact support, $\exists N \geq 0$, $\exists C > 0$ s.t.
\[
\forall f \in C_c^\infty(\mathbb{R}), \quad |\langle f, \mu \rangle| \leq C \|f\|_N = C \max_{n=0, \ldots, N} \|f^{(n)}\|_{\infty}.
\]

Let
\[K = \bigcap_{n=0}^{N} \ker (D^n f).
\]
We will show that
\[f \in K \Rightarrow \langle f, \mu \rangle = 0.\]

It then follows from a preceding lemma that
\[\mu = \sum_{k=0}^{N} c_n D^n f\] for some scalar $c_n$. 

\[\blacksquare\]
These scalars cannot all be zero, for if they were then we would have \( \mu = 0 \) and \( \text{supp}(\mu) = \emptyset \). Hence to proof a complete once we show that (x) holds.

So, suppose \( f \in \mathcal{K} \), i.e., \( f^{(m)}(0) = 0 \) for \( n = 0, \ldots, N \). Choose any \( \gamma > 0 \). By a preceding lemma, \( \exists \gamma > 0 \) s.t.

\[
|x| < r, \quad n = 0, \ldots, N \quad \Rightarrow \quad |f^{(m)}(x)| \leq \gamma |x|^{N-n}.
\]

WLOG, assume \( r < 1 \).

Let \( \theta \in C_c^\infty(\mathbb{R}) \) be such that \( \text{supp}(\theta) \subseteq (-1,1) \) and \( \theta = 1 \) on \([ -\frac{1}{2}, \frac{1}{2} ] \).

Set \( \theta_r(x) = \theta(x/r) \), and note that

\[
\| \theta_r^{(j)} \|_\infty = \frac{1}{r^j} \| \theta^{(j)} \|_\infty, \quad j \geq 0.
\]

Then for \( |x| < r \) and \( n = 0, \ldots, N \) we have

\[
| \sum \langle \theta_r^{(j)} \eta^{(j)}(t), f^{(j)}(t) \rangle \rangle \leq |1| = 1,
\]
\[ |(f \bar{\theta}_e)^{(n)}(x)| \leq \sum_{j=0}^{n} \binom{n}{j} |f^{(j)}(x)| |\theta^{(n-j)}(x)| \]

\[ \leq \sum_{j=0}^{n} \binom{n}{j} \eta |x|^{n-j} \frac{1}{\eta^{n-j}} \|\theta^{(n-j)}\|_{\infty} \]

\[ \leq \eta \sum_{j=0}^{n} \binom{n}{j} \|\theta^{(n-j)}\|_{\infty} \]

\[ = C'_n \eta, \]

where \( C'_n = \sum_{j=0}^{n} \binom{n}{j} \|\theta^{(n-j)}\|_{\infty} \) is a constant independent of \( \eta \). Since \( \text{supp}(f \bar{\theta}_e) \subseteq (-\delta, \gamma) \), this implies that

\[ \| (f \bar{\theta}_e)^{(n)} \|_{\infty} \leq C'_n \eta, \quad n = 0, \ldots, N. \]

Now, since \( \theta_r = 1 \) on a neighborhood of \( \text{supp}(\mu) \), we have \( \theta \mu = \mu \) by an earlier exercise. Consequently,
\[ |\langle f, \mu \rangle | = |\langle f, \theta \mu \rangle | \]
\[ = |\langle f \theta, \mu \rangle | \]
\[ \leq C \| f \theta \|_N \]
\[ \leq C \sum_{n=0}^{N} \| (f \theta)_n \|_\infty \]
\[ \leq \eta C \sum_{n=0}^{N} C_n^\prime. \]

Since \( \eta \) is arbitrary, this implies \( \langle f, \mu \rangle = 0 \), so (x) holds and the proof is complete. \( \Box \)

**Exercise**

Suppose \( \theta + \mu \in \Omega(\mathbb{R}) \). Show that

\[ f = 0 \text{ on } \text{supp}(\mu) \implies \langle f, \mu \rangle = 0. \]

The following exercises will provide conditions on \( f \) that do imply that \( \langle f, \mu \rangle = 0 \).
Exercise (see Rudin, F.A., 2nd ed., p.179).

Let $\mu \in \mathcal{E}'(\mathbb{R}) = C^\infty(\mathbb{R})^*$ and $f \in C^\infty(\mathbb{R})$ be given. Let $N$ be the order of $\mu$. Show that if $f^{(n)} = 0$ on supp($\mu$) for $n = 0, \ldots, N$, then $\langle f, \mu \rangle = 0$.

Hint: Follow the same techniques used in the preceding theorem & lemma. Also, supp($\mu$) is compact - every open cover has a finite subcover.

Exercise (see Rudin, p.179)

Let $\mu \in \mathcal{E}'(\mathbb{R})$ & $f \in C_c^\infty(\mathbb{R})$ be given. Show that if $f^{(n)} = 0$ on supp($\mu$) for $n \geq 0$, then $\langle f, \mu \rangle = 0$.

Hint: Choose a smooth cutoff function $\theta$. 
We can use the preceding exercises to give another proof that a distribution supported on a single point is a finite linear combination of δ distributions at that point.

**Theorem**

Given \( \mu \in \mathcal{D}'(\mathbb{R}) \), TFAE:

a. \( \text{supp}(\mu) = \{0\} \),

b. \( \exists N \geq 0, c_0, \ldots, c_N \in \mathbb{C} \) not all zero s.t.

\[
\mu = \sum_{n=0}^{N} c_n \delta^n.
\]

**Proof**

\( b \Rightarrow a \), Exercise.

\( a \Rightarrow b \). Suppose \( \text{supp}(\mu) = \{0\} \),

By a previous theorem, since \( \mu \) has compact support it has finite order, say order \( N \). Since \( \mu \in \mathcal{E}'(\mathbb{R}) = \mathcal{E}(\mathbb{R})' \), fix any \( f \in C^\infty(\mathbb{R}) \).

\( \langle X^n, \mu \rangle \) exists for all \( n \geq 0 \). Since \( f \) is \( C^\infty \), we can write it in a Taylor expansion as
\[ f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n + r(x) \]

where \( r \in C^\infty(\mathbb{R}) \). Further, for \( n = 0, \ldots, N \) we have

\[
r^{(m)}(0) = f^{(m)}(0) - \frac{d^m}{dx^m} \left( \sum_{k=0}^{N} \frac{f^{(k)}(0)}{n!} x^k \right)(0)
\]

\[ = f^{(m)}(0) - f^{(n)}(0) \]

\[ = 0. \]

Hence by the preceding exercise

\[ \langle r, \mu \rangle = 0, \]

so

\[ \langle f, \mu \rangle = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} \langle x^n, \mu \rangle \]

\[ = \sum_{n=0}^{N} \frac{(-1)^n \langle x^n, \mu \rangle}{n!} \langle f, \mathcal{D}^n \mu \rangle \]

\[ = \langle f, \sum_{n=0}^{N} \frac{(-1)^n \langle x^n, \mu \rangle}{n!} \mathcal{D}^n \mu \rangle. \]

Thus

\[ \mu = \sum_{n=0}^{N} \frac{(-1)^n \langle x^n, \mu \rangle}{n!} \mathcal{D}^n \mu. \]