3.6 Tempered Distribution

Since $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$, we know that the F.T. maps $C_c^\infty(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$. However, by the Paley-Wiener theorem, if $f$ has compact support then $\hat{f}$ cannot be compactly supported. Hence the F.T. does not map $C_c^\infty(\mathbb{R})$ into itself. A consequence of this is that, unlike such operators as translation, modulation, dilation, differentiation, we cannot extend the F.T. to all of $C_c^\infty(\mathbb{R})^* = \mathcal{D}'(\mathbb{R})$.

Instead, we must restrict the extension to a space properly contained in $\mathcal{D}'(\mathbb{R})$, namely $\mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$, the space of tempered distributions.

To emphasize this point, if we extend the definition of the F.T., then this definition should extend the existing functional properties of the F.T. In
particular, if \( f, g \in L^2(\mathbb{R}) \) then we have Parseval equality:
\[
\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle
\]
Restricting to \( f \in \mathcal{S}(\mathbb{R}) \), since \( \mathcal{F} \) is an F.T. map \( \mathcal{S}(\mathbb{R}) \) onto itself, it implies that if \( g \in L^2(\mathbb{R}) \) then
\[
\forall f \in \mathcal{S}(\mathbb{R}), \quad \langle f, \hat{g} \rangle = \langle f, \hat{g} \rangle.
\]
This property tells us how to extend \( \mathcal{F} \) F.T. to elements in \( \mathcal{S}'(\mathbb{R}) \): if \( \mu \in \mathcal{S}'(\mathbb{R}) \) then \( \hat{\mu} \) is the element of \( \mathcal{S}'(\mathbb{R}) \) defined by
\[
\forall f \in \mathcal{S}(\mathbb{R}), \quad \langle f, \hat{\mu} \rangle = \langle f, \mu \rangle.
\]
Of course, we must demonstrate that \( \hat{\mu} \) does define an element of \( \mathcal{S}'(\mathbb{R}) \), & to do that we must first review the meaning of continuity of a linear functional \( \mu : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C} \), or, in equivalent terms, \( \mu \).
definition of $\mathcal{L}$-topology on $\mathcal{L}(\mathbb{R})$.

**Definition**
Given $f_k, g \in \mathcal{L}(\mathbb{R})$, we say that $f_k \to g$ in $\mathcal{L}(\mathbb{R})$ if

$$\forall m, n \geq 0, \quad \| x^m g^{(n)}(x) - x^m f_k^{(n)}(x) \|_\infty \to 0$$

as $k \to \infty$.

**Definition**

a. A linear functional $\mu : \mathcal{L}(\mathbb{R}) \to \mathbb{C}$ is **continuous** if

$$f_k \to 0 \text{ in } \mathcal{L}(\mathbb{R}) \implies \langle f_k, \mu \rangle \to 0.$$

b. The space of **tempered distributions** is

$$\mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^* = \{ \mu : \mathcal{S}(\mathbb{R}) \to \mathbb{C} : \mu \text{ is a continuous linear functional} \}$$

**Exercise**
Prove that $\delta \in \mathcal{S}'(\mathbb{R})$.

**Exercise**
Show that if $f_k, f \in \mathcal{L}(\mathbb{R})$ & $f_k \to f$ in $\mathcal{L}(\mathbb{R})$, then $f'_k \to f'$ in $\mathcal{L}(\mathbb{R})$. 

Example
Recall that $L^1_{\text{loc}}(\mathbb{R}) \subseteq \mathcal{O}'(\mathbb{R})$. In particular, $g(x) = e^{-x^2} \in L^1_{\text{loc}}(\mathbb{R}) \subseteq \mathcal{O}'(\mathbb{R})$. However, if we take $f(x) = e^{-x^2} \in \mathcal{S}'(\mathbb{R})$, then
\[
\langle f, g \rangle = \int e^{-x^2} e^{x^2} \, dx
\]
does not even converge. Hence $g$ does not define an element of $\mathcal{S}'(\mathbb{R})$.

Exercise
Prove that if $g \in L^1_{\text{loc}}(\mathbb{R})$ and
\[
\exists \text{ polynomial } p \text{ s.t. } \ |g(x)| \leq |p(x)| \quad \forall x \text{ large enough},
\]
then $g \in \mathcal{D}'(\mathbb{R})$.

Thus $\mathcal{D}'(\mathbb{R})$ includes all locally integrable functions with polynomial growth at $\infty$. However, the requirement of polynomial growth is only sufficient.
not necessary, as the following example shows.

**Example:** We have $g(x) = \sin e^x \in L^\infty(\mathbb{R}) = \mathcal{S}'(\mathbb{R})$. Since $f \in C^\infty(\mathbb{R})$, its distributional derivative coincides with its pointwise derivative, so we know that

$$g'(x) = e^x \cos e^x \in C^\infty(\mathbb{R}).$$

Thus does not exist any polynomial $p$ that dominates $g'$.

Yet if $f \in \mathcal{S}(\mathbb{R})$ then

$$\lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b f(x) e^x \cos e^x \, dx$$

$$= \lim_{a \to -\infty} \lim_{b \to \infty} \left[ f(b) \frac{\sin b}{b} - f(a) \sin a - \int_a^b f'(x) \sin e^x \, dx \right]$$

$$= -\int_{-\infty}^\infty f'(x) \sin e^x \, dx$$

$$= -\langle f', g \rangle.$$

Hence $\langle f, g \rangle = \int_{-\infty}^\infty f(x) e^x \cos e^x \, dx = 0 - \langle f', g \rangle$ well-defined. Further if $f_k \to f$ in $\mathcal{S}(\mathbb{R})$ then
since $f_e \to f$ in $\mathcal{D}''(\mathbb{R})$, we have that

$$\langle f_e, g' \rangle = -\langle f_e', g \rangle \to -\langle f', g \rangle = \langle f, g' \rangle.$$

Hence $g' \in \mathcal{D}''(\mathbb{R})$.

Remark (see Folland, Real Analysis, 2nd ed., p. 294)

Although $f'$ is not bounded by a polynomial, “on average” it is not “too large”, because of its rapid oscillations.

Exercise

$e^{\pm ix} \notin \mathcal{D}'(\mathbb{R})$. 
Exercise
Show that for each $1 \leq p \leq \infty$, $L^p(\mathbb{R}) \subseteq L^1(\mathbb{R})$.

Note that if $f \in L^p(\mathbb{R})$ with $p < \infty$, then $f$
need not have polynomial growth at $\infty$. 
Remark

$\mathcal{S}(\mathbb{R})$ is a topological vector space whose topology is defined by a countably family of seminorms

$$p_{mn}(f) = \|x^m f^{(n)}(x)\|_\infty, \quad m, n \geq 0.$$  

If we define "balls"

$$B_{\varepsilon}^{mn}(f) = \{ g \in \mathcal{S}(\mathbb{R}) : p_{mn}(f - g) < \varepsilon \},$$

then the topology on $\mathcal{S}(\mathbb{R})$ is generated by these "balls".

More precisely, a collection $\mathcal{B}$ of all finite intersections of the $B_{\varepsilon}^{mn}(f)$ is a base for the topology, meaning every open set is a union of base elements. Consequently, if $U \subseteq \mathcal{S}(\mathbb{R})$ is open & $f \in U$, then $\exists$ finitely many $m_j, n_j, \varepsilon_j$ s.t.

$$\bigcap_{j=1}^{N} B_{\varepsilon_j}^{m_j, n_j}(f) \subseteq U.$$  

Taking as a definition that $\mu: \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is continuous if $\mu^{-1}(V)$ is open in $\mathcal{S}(\mathbb{R})$ for each open $V \subseteq \mathbb{C}$,
it is shown in Appendix that our definition of continuity is a consequence.

Because of the following two facts:

- There are countably many seminorms,
- The topology is Hausdorff, which is equivalent to

\[ \beta_n(f) = 0 \quad \forall m, n \geq 0 \implies f = 0, \]

we can create a metric on \( \mathcal{S}(\mathbb{R}) \) that generates the same topology, namely,

\[ d(f, g) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} \frac{\beta_n(f - g)}{1 + \beta_n(f - g)}. \]

That is, \( f_k \to f \) in \( \mathcal{S}(\mathbb{R}) \) if and only if \( d(f_k, f) \to 0 \).

Consequently, the notion of Cauchy sequences in \( \mathcal{S}(\mathbb{R}) \) makes sense, & we can ask if \( \mathcal{S}(\mathbb{R}) \) is complete, i.e., is every Cauchy sequence convergent?

In fact it is (exercise), so \( \mathcal{S}(\mathbb{R}) \) is a vector space that
is a complete metric space. Such a space is called a Frechet space. Unfortunately, $\mathbb{L}$ metric on $\mathbb{L}(\mathbb{R})$ is not induced by any norm, so $\mathbb{L}(\mathbb{R})$ is not a Banach space.

Expanded discussion of these issues is given in the Appendix.
Remark
Because the topology on $\mathcal{S}(\mathbb{R})$ is given by a family of seminorms, there is an analogue of "continuity = boundedness." The following is a consequence of a result proved in Chapter 2 Appendix.

Theorem
Let $\mu : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ be a linear functional. Then TFAE:

a. $\mu$ is continuous, i.e., $\mu \in \mathcal{L}(\mathcal{S}(\mathbb{R}))$,

b. $\exists M, N \geq 0$, $\exists C < \infty$ s.t.

$$\forall f \in \mathcal{S}(\mathbb{R}), \quad |\langle f, \mu \rangle| \leq C \sum_{m=0}^{M} \sum_{n=0}^{N} \| x^m \phi_n^{(n)} \|_{\infty}$$
Definition

The topology on $\mathcal{D}'(\mathbb{R})$ is the weak* topology, i.e., if $\mu_n, \mu \in \mathcal{D}'(\mathbb{R})$, then $\mu_n \to \mu$ in $\mathcal{D}'(\mathbb{R})$ means

$$\forall f \in \mathcal{D}(\mathbb{R}), \quad \langle f, \mu_n \rangle \to \langle f, \mu \rangle.$$

Exercise

Suppose that $\mu_n \to \mu$ in $\mathcal{D}'(\mathbb{R})$ & $E$ compact. Let $K \subseteq \mathbb{R}$ s.t. $\text{supp}(\mu_n) \subseteq K$ $\forall n$. Show that $\text{supp}(\mu) \subseteq K$. 
Exercise
Suppose that \( f_k, f \in L^1(\mathbb{R}) \) \& \( f_k \to f \) in \( L^1(\mathbb{R}) \).

a. Prove that if \( g \in L^1(\mathbb{R}) \) then \( f_k \cdot g \to f \cdot g \) in \( L^1(\mathbb{R}) \).

b. Prove that if \( g \in L^1(\mathbb{R}) \) \& \( f_k \ast g \to f \ast g \) in \( L^1(\mathbb{R}) \).

Hint: \((f \ast g)^m = f^m \ast g \) and 
\[ x^m = (x - y + y)^m = \sum_{j=0}^{m} \binom{m}{j} (x-y)^j y^{m-j} . \]
is a complete metric space. Such a space is called a Fréchet space. Unfortunately, the metric on $L^1(\mathbb{R})$ is not necessarily induced from any norm, so $L^1(\mathbb{R})$ is not a Banach space.

Expanded discussion of these issues is given in the Appendix.

\begin{lemma}
$\mathcal{C}_c^\infty(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, in the topology of $L^1(\mathbb{R})$. That is, if $f \in L^1(\mathbb{R})$, then there exists $f_k \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $f_k \to f$ in $L^1(\mathbb{R})$.
\end{lemma}

\begin{proof}
Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ be any fixed function such that $\text{supp}(\varphi) \subseteq [-1, 1]$ and $\varphi(0) = 1$. Define $\varphi_k(x) = \varphi(x/k)$.
Then $f\theta_k \in C^\infty(\mathbb{R})$ & we claim that $f\theta_k \to f$ in $\mathcal{D}'(\mathbb{R})$.

To prove this, we must show that

$$\forall m,n \geq 0, \quad \| x^m f^{(n)}(x) - x^m (f\theta_k)^{(n)}(x) \|_\infty \to 0$$

as $k \to \infty$.

**Case 1: $n = 0$**

Fix any $m \geq 0$ & any $\varepsilon > 0$. Then $\exists k_0 \geq 0$ s.t.

$$\sup_{|x| \geq k_0} |x^m f(x)| < \varepsilon.$$  

Hence for $k > k_0$, we have that

$$\| x^m f(x) - x^m (f\theta_k)(x) \|_\infty \leq \sup_{|x| > k} |x^m f(x)| < \varepsilon.$$  

Thus $\| x^m f(x) - x^m (f\theta_k)(x) \|_\infty \to 0$ as $k \to \infty$.

**Case 2: $n > 0$**

Fix any $n > 0 \& m \geq 0$. Then by the product rule for derivatives, we have that
\[ \| x^m f^{(m)}(x) - x^m (f \Theta_k)^{(m)}(x) \|_\infty \]
\[ = \| x^m f^{(m)}(x) - x^m \sum_{j=0}^{n-1} \binom{n}{j} f^{(j)}(x) \Theta_k^{(n-j)}(x) \|_\infty \]
\[ \leq \| x^m f^{(m)}(x) - x^m f^{(m)}(x) \Theta_k(x) \|_\infty \]
\[ + \sum_{j=0}^{n-1} \binom{n}{j} \| x^m f^{(j)}(x) \Theta_k^{(n-j)}(x) \|_\infty \]
\[ = I_1(k) + I_2(k). \]

Now, since \( f^{(m)} \in \mathcal{S}(\mathbb{R}) \), we have by Case 1 that
\[ I_1(k) \rightarrow 0 \text{ as } k \rightarrow \infty. \]
Also,
\[ \Theta_k^{(j)}(x) = \frac{d^j}{dx^j} \Theta_k(x) = \frac{1}{k^j} \Theta^{(j)}(x). \]
So,
\[ I_2(k) \leq \sum_{j=0}^{n-1} \binom{n}{j} \| x^m f^{(j)}(x) \|_\infty \| \Theta_k^{(n-j)} \|_\infty \]
\[ \frac{\| \Theta_k^{(n-j)} \|_\infty}{k^{n-j}} \]
\[ \rightarrow 0 \text{ as } k \rightarrow \infty. \]
\[
\text{since } f' \in \mathcal{S}(\mathbb{R}), \text{ we have } \mathcal{D} \\
\langle f'_*, g' \rangle = -\langle f'_*, g \rangle = -\langle f'_*, g \rangle = \langle f'_*, g' \rangle \\
\text{Hence } g' \in \mathcal{S}(\mathbb{R}).
\]

Remark (See Folland, Real Analysis, 2nd ed., p. 294)

Although \( f' \) is not bounded by a polynomial, "on average" it is not "too large" because of its rapid oscillations.

Motivation

Since \( \mathcal{C}_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}) \), we may expect that
\( \mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^* \subseteq \mathcal{C}_c^\infty(\mathbb{R})^* = \mathcal{D}'(\mathbb{R}) \), since it should be "easier" for a linear functional to be continuous on a smaller domain than a larger one.

However, to make it precise \( \mathcal{S}'(\mathbb{R}) \) must be a connection between convergence in \( \mathcal{C}_c^\infty(\mathbb{R}) \) & convergence in \( \mathcal{S}(\mathbb{R}) \), or in other words, a connection between their respective topologies.
Exercise
Show that if $f_k, f \in C_c^\infty(\mathbb{R})$ then

$$f_k \to f \text{ in } C_c^\infty(\mathbb{R}) \implies f_k \to f \text{ in } \mathcal{D}(\mathbb{R}).$$

Theorem
$\mathcal{D}'(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$. More precisely,

$$\mu \mapsto \mu|_{C_c^\infty(\mathbb{R})}$$

is a continuous injection of $\mathcal{D}'(\mathbb{R})$ into a proper subspace of $\mathcal{D}'(\mathbb{R})$.

Proof:
Suppose $\mu \in \mathcal{D}'(\mathbb{R})$. Then $\mu|_{C_c^\infty(\mathbb{R})} : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ is a linear functional. Further, if $f_k \to f$ in $C_c^\infty(\mathbb{R})$, then by the exercise we have $f_k \to f$ in $\mathcal{D}(\mathbb{R})$, so

$$\langle f_k, \mu \rangle \to \langle f, \mu \rangle \text{ since } \mu \text{ is continuous on } \mathcal{D}(\mathbb{R}).$$

Thus $\mu|_{C_c^\infty(\mathbb{R})} \in \mathcal{D}'(\mathbb{R})$.

Thus $\mathcal{U} : \mu \mapsto \mu|_{C_c^\infty(\mathbb{R})}$ is a well-defined, antilinear map. To show it is injective, we need only
show \( \ker(U) = \{0\} \). So, suppose \( \mu \in \mathfrak{S}'(\mathbb{R}) \) and \\
\(\mu|_{C^\infty(\mathbb{R})} = 0 \). Fix any \( f \in \mathfrak{S}(\mathbb{R}) \). By an earlier theorem, \( C^\infty(\mathbb{R}) \) is dense in \( \mathfrak{S}(\mathbb{R}) \), so \\
\exists f_k \in C^\infty(\mathbb{R}) \text{ s.t. } f_k \to f \text{ in } \mathfrak{S}(\mathbb{R}) \).

Then, since \( \mu \) is continuous, \( f_k \in \ker(\mu|_{C^\infty(\mathbb{R})}) \), \\
\(0 = \langle f_k, \mu \rangle \to \langle f, \mu \rangle,
\)

Thus \( \langle f, \mu \rangle = 0 \) \( \forall f \in \mathfrak{S}(\mathbb{R}) \), so \( \mu = 0 \).

Next, to show \( U \) is continuous, suppose that \\
\( \mu_n \to \mu \) in \( \mathfrak{S}'(\mathbb{R}) \). Since convergence in \( \mathfrak{S}'(\mathbb{R}) \) is weak* convergence, we have \\
\( \forall f \in \mathfrak{S}(\mathbb{R}), \quad \langle f, \mu_n \rangle \to \langle f, \mu \rangle \).

But then \( \mathfrak{S} \) same is true \( \forall f \in C^\infty(\mathbb{R}) \), so \\
\( \mu_n|_{C^\infty(\mathbb{R})} \to \mu|_{C^\infty(\mathbb{R})} \). Thus \( U \) is continuous.

Finally, we already gave an example of a
distribution that belongs to $\mathcal{D}'(\mathbb{R})$ but not $\mathcal{S}'(\mathbb{R})$. Hence $U$ is not surjective.

Remark (see Benedetto, p. 92) The Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re}(s) > 1.$$ 

It has an analytic continuation to $\mathbb{C} \setminus \{1\}$, with a simple pole at $s = 1$. The Riemann hypothesis, whose validity is the most celebrated open problem in analytic number theory, is a statement that all complex zeros of $\zeta(s)$ can only occur when $\text{Re}(s) = \frac{1}{2}$.

The Weil distribution $\text{WE} \mathcal{D}'(\mathbb{R})$ is defined in [Benedetto, Harmonic Analysis & Applications, p. 92].

Benedetto has shown that:

The Riemann Hypothesis is valid $\iff \text{WE} \mathcal{S}'(\mathbb{R})$. 
Exercise

Show that $\mathcal{E}'(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$. Recall that $\mathcal{E}'(\mathbb{R})$ is the space of compactly supported distributions, as shown in a previous theorem to equal $\mathcal{C}^\infty(\mathbb{R})^\times$. Show that $\mu \mapsto \mu/\|\mu\|_{\mathcal{S}'(\mathbb{R})}$ is a continuous injection of $\mathcal{E}'(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$ that is not surjective.
Exercise
Let $\mu \in D'(\mathbb{R})$ and $f \in D(\mathbb{R})$ be given.

a. Show that $f * \mu \in C(\mathbb{R})$.

Hint: Show $\frac{1}{\alpha} f \to f$ in $D'(\mathbb{R})$ as $\alpha \to 0$.

b. Show $f * \mu \in C'(\mathbb{R})$, and $(f * \mu)' = f' * \mu$.

Hint: Show $\frac{1}{\alpha} f \to f'$ in $D'(\mathbb{R})$ as $\alpha \to 0$.

c. Show $f * \mu \in C^\infty(\mathbb{R})$.

Remark
$C^\infty(\mathbb{R})$ is not contained in $D'(\mathbb{R})$, e.g.,
$e^{x^2} \notin D'(\mathbb{R})$. However, the next result will show that $f * \mu \in D'(\mathbb{R})$. 

Theorem

If \( f \in L^1(\mathbb{R}) \) and \( \mu \in L^2(\mathbb{R}) \) then \( f \ast \mu \in C^0(\mathbb{R}) \) with at most polynomial growth at \( \infty \). Hence \( f \ast \mu \in L^1(\mathbb{R}) \).

Proof:

By an earlier result, \( \exists M, N > 0 \) and \( C > 0 \) s.t.

\[ \forall \phi \in \mathcal{S}(\mathbb{R}), \quad |\langle \phi, \mu \rangle| \leq C \sum_{m=0}^{M} \sum_{n=0}^{N} \| \lambda^m (\phi^{(n)}(\cdot)) \|_\infty. \]

Hence,

\[ |(f \ast \mu)(y)| = |\langle \frac{\lambda^m}{\lambda^M}, \mu \rangle| \]

\[ \leq C \sum_{m=0}^{M} \sum_{n=0}^{N} \| \lambda^m (T_{\lambda^M} \phi^{(n)}(\cdot)) \|_\infty \]

\[ = C \sum_{m=0}^{M} \sum_{n=0}^{N} \| \lambda^m \phi^{(n)}(y-x) \|_\infty \]

\[ = C \sum_{m=0}^{M} \sum_{n=0}^{N} \| (y-x)^m \phi^{(n)}(\cdot) \|_\infty \]

\[ \leq C \sum_{m=0}^{M} \sum_{j=0}^{n} \sum_{j=0}^{m} |y|^{m-j} \| \lambda^j \phi^{(n)}(\cdot) \|_\infty \]

This is a polynomial in \( |y| \), so \( f \ast \mu \) has polynomial growth.
Exercise
Show that if \( f \in L^1(\mathbb{R}) \) and \( \mu \in \mathcal{D}'(\mathbb{R}) \), then \( f * \mu \) need not be integrable.

Hint: \( e^{-x^2} * x^2 \).

Exercise
Given \( \mu \in \mathcal{D}'(\mathbb{R}) \), prove that \( L : \mathcal{D}(\mathbb{R}) \to C^\infty(\mathbb{R}) \),
\[ f \mapsto f * \mu \]
is a linear & continuous map.