B

Lebesgue Measure and Integral

In this volume, measures make an appearance in essentially two distinct ways. First, Lebesgue measure on $\mathbb{R}$ (and sometimes on $\mathbb{R}^2$ or $\mathbb{R}^d$), is used throughout to define the Fourier transform and the spaces that it acts upon, such as the Lebesgue spaces $L^p(\mathbb{R})$. Second, the class $M_b(\mathbb{R})$ of bounded Borel measures enters in Chapter 4 as another space of objects upon which the Fourier transform acts.

Since Lebesgue measure is a particular unbounded Borel measure, if our goal was conciseness then it would make sense to develop abstract measure theory with Lebesgue measure as one special case of this theory. However, since the role of Lebesgue measure in this volume is so fundamental and because there are many facts that we will need that are specific to Lebesgue measure, we choose to review the theory of Lebesgue measure and integration here as a topic in itself, and to separately review the theory of general Borel measures in Appendix D.

For more detail on this material, we refer to the texts by Wheeden and Zygmund [WZ77] and Folland [Fol99]. Additionally, more details on the Banach–Zarecki Theorem can be found in the texts by Benedetto [Ben76] or Bruckner, Bruckner, and Thomson [BBT97].

B.1 Exterior Lebesgue Measure

We begin with the familiar notion of the volume of a rectangular box in $\mathbb{R}^d$, which for simplicity we refer to as a "cube" (even though we do not require all side lengths to be equal).

Definition B.1. A *cube* in $\mathbb{R}^d$ is a set of the form

$$Q \;=\; [a_1,b_1]\times\cdots\times[a_d,b_d] \;=\; \prod_{i=1}^{d}[a_i,b_i].$$

The *volume* of this cube is

$$\mathrm{vol}(Q) \;=\; (b_1 - a_1) \cdots (b_d - a_d) \;=\; \prod_{i=1}^{d} (b_i - a_i).$$

We extend the notion of volume to arbitrary sets by covering them with countably many cubes in all possible ways.

Definition B.2. The *exterior Lebesgue measure* or *outer Lebesgue measure* of a set $E \subseteq \mathbb{R}^d$ is

$$|E|_e \;=\; \inf\Bigl\{\sum_k \mathrm{vol}(Q_k)\Bigr\}$$

where the infimum is taken over all *finite or countable* collections of cubes Q_k such that $E \subseteq \bigcup_k Q_k$.

Thus, every subset of $\mathbb{R}^d$ has a uniquely defined exterior measure, which lies in the range $0 \le |E|_e \le \infty$.

There are many seemingly "obvious" results concerning Lebesgue measure that actually require rather tedious proofs, many of which we will omit. For example, although it is clear from the definition that $|Q|_e \le \mathrm{vol}(Q)$, it requires some care to show that the exterior measure of a cube actually coincides with its volume. We state this now without proof.

Theorem B.3. (a) *If Q is a cube in $\mathbb{R}^d$ then $|Q|_e = \mathrm{vol}(Q)$.*
(b) *If $Q_1, \dots, Q_n$ are disjoint cubes in $\mathbb{R}^d$, then*

$$\Bigl|\bigcup_{k=1}^{n} Q_k\Bigr|_e \;=\; \sum_{k=1}^{n} \mathrm{vol}(Q_k).$$

The next exercise gives some basic properties of exterior measure.

Exercise B.4. (a) Monotonicity: If $E \subseteq F \subseteq \mathbb{R}^d$, then $|E|_e \le |F|_e$.
(b) Countable subadditivity: If $E_k \subseteq \mathbb{R}^d$ for $k \in \mathbb{N}$, then

$$\Bigl|\bigcup_{k=1}^{\infty} E_k\Bigr|_e \;\le\; \sum_{k=1}^{\infty} |E_k|_e.$$

(c) Translation invariance: If $E \subseteq \mathbb{R}^d$ and $h \in \mathbb{R}^d$, then $|E+h|_e = |E|_e$, where $E + h = \{x + h : x \in E\}$.

Our next theorem gives a type of "regularity" property for exterior Lebesgue measure: Every set E can be surrounded by an *open* set U whose exterior measure is only ε larger than that of E (by monotonicity we also have $|E|_e \le |U|_e$, so the measure of U is very close to the measure of E).

Theorem B.5. *If $E \subseteq \mathbb{R}^d$ and $\varepsilon > 0$, then there exists an open set $U \supseteq E$ such that $|U|_e \le |E|_e + \varepsilon$, and hence*

$$|E|_e \;=\; \inf\bigl\{|U|_e : U \text{ open, } U \supseteq E\bigr\}. \tag{B.1}$$

Proof. If $|E|_e = \infty$, take $U = \mathbb{R}^d$. Otherwise we have $|E|_e < \infty$, so by definition of exterior measure there must exist cubes Q_k such that $E \subseteq \cup Q_k$ and $\sum \text{vol}(Q_k) < |E|_e + \frac{\varepsilon}{2}$. Let Q_k^* be a larger cube that contains Q_k in its interior, and such that $\text{vol}(Q_k^*) \le \text{vol}(Q_k) + 2^{-k-1}\varepsilon$. Let U be the union of the interiors of the cubes Q_k^*. Then $E \subseteq U$, U is open, and

$$|U|_e \;\le\; \sum_k \text{vol}(Q_k^*) \;\le\; \sum_k \text{vol}(Q_k) \;+\; \frac{\varepsilon}{2} \;<\; |E|_e + \varepsilon. \qquad \square$$

Since E and $U \backslash E$ are disjoint and their union is U, we might expect that the sum of their exterior measures is the exterior measure of U. Unfortunately, this is false in general (although the Axiom of Choice is required to show the existence of a counterexample, see Problem 1.20). Consequently, the fact that $|U|_e \le |E|_e + \varepsilon$ does *not* imply that $|U\backslash E|_e \le \varepsilon$! The "well-behaved" sets for which this is true will be said to be *measurable*, and will be studied next.

B.2 Lebesgue Measure

B.2.1 Definition and Basic Properties

Definition B.6. A set $E \subseteq \mathbb{R}^d$ is *Lebesgue measurable*, or simply *measurable*, if

$$\forall\, \varepsilon > 0, \quad \exists \text{ open } U \supseteq E \text{ such that } |U\backslash E|_e \le \varepsilon.$$

If E is Lebesgue measurable, then its *Lebesgue measure* is $|E| = |E|_e$. We set

$$\mathcal{L} \;=\; \mathcal{L}(\mathbb{R}^d) \;=\; \{E \subseteq \mathbb{R}^d : E \text{ is Lebesgue measurable}\}.$$

Thus, there is no difference between the numeric value of the Lebesgue measure and the exterior Lebesgue measure of a measurable set, but when we know that E is measurable we denote this value using the symbols $|E|$ instead of $|E|_e$.

Exercise B.7. Show that if $|E|_e = 0$, then $E \in \mathcal{L}$.

Consequently, if $|E|_e = 0$, then not only is E measurable, but every subset of E is measurable. In the language of abstract measure theory, the measure space $(\mathbb{R}^d, \mathcal{L}, |\cdot|)$ is said to be *complete*.

The following result, whose proof will be omitted, summarizes some of the basic properties of measurable sets.

Theorem B.8. (a) *$\mathcal{L}$ is a σ-algebra of subsets of $\mathbb{R}^d$. That is:*

i. *$\emptyset$, $\mathbb{R}^d \in \mathcal{L}$,*

ii. *if $E_1, E_2, \dots \in \mathcal{L}$, then $\cup E_k \in \mathcal{L}$,*

iii. *if $E \in \mathcal{L}$, then $\mathbb{R}^d \backslash E \in \mathcal{L}$.*

(b) *Every open and every closed set belongs to $\mathcal{L}$.*

Since $\mathcal{L}$ is closed under complements and countable unions, it follows that it is closed under countable intersections as well. General σ-algebras are discussed in more detail in Section D.1.

Since exterior Lebesgue measure is subadditive, the same is true for Lebesgue measurable sets, i.e., if E_k for $k \in \mathbb{N}$ are measurable subsets of $\mathbb{R}^d$ then we have $\left|\cup E_k\right| \leq \sum |E_k|$. Although we will not prove it, one of the main reasons for restricting to Lebesgue measurable sets is that for these sets we have *additivity* of the Lebesgue measure of disjoint measurable sets. The analogous statement for exterior Lebesgue measure does not hold in general!

Theorem B.9 (Countable Additivity of Lebesgue Measure). *If E_1, $E_2, \dots$ are disjoint Lebesgue measurable subsets of $\mathbb{R}^d$, then*

$$\left|\bigcup_{k=1}^{\infty} E_k\right| = \sum_{k=1}^{\infty} |E_k|.$$

Here are some additional basic properties of Lebesgue measure.

Exercise B.10. Let E and E_k for $k \in \mathbb{N}$ be measurable subsets of $\mathbb{R}^d$, and prove the following facts.

(a) If $E_1 \subseteq E_2$ and $|E_1| < \infty$, then $|E_2 \backslash E_1| = |E_2| - |E_1|$.

(b) Continuity from below: If $E_1 \subseteq E_2 \subseteq \cdots$, then $\left|\cup E_k\right| = \lim_{k\to\infty} |E_k|$.

(c) Continuity from above: If $E_1 \supseteq E_2 \supseteq \cdots$ and $|E_1| < \infty$, then $\left|\cap E_k\right| = \lim_{k\to\infty} |E_k|$.

(d) Translation invariance: If $h \in \mathbb{R}^d$, then $|E + h| = |E|$, where $E + h = \{x + h : x \in E\}$.

When a linear change of variable is made, the measure of a set is multiplied by the absolute value of the determinant of the transformation. Linear transformations are special cases of *Lipschitz functions*, compare Definition B.71 and Problem B.22.

Theorem B.11. *If $T\colon \mathbb{R}^d \to \mathbb{R}^d$ is linear and $E \subseteq \mathbb{R}^d$ is Lebesgue measurable, then $T(E)$ is measurable and $|T(E)| = |\det(T)|\,|E|$.*

B.2.2 Equivalent Formulations of Measurability

Definition B.12. (a) A set $H \subseteq \mathbb{R}^d$ is a *G_δ-set* if there exist finitely or countably many open sets U_k such that $H = \cap U_k$.

(b) A set $H \subseteq \mathbb{R}^d$ is an *F_σ-set* if there exist finitely or countably many closed sets F_k such that $H = \cup F_k$.

Exercise B.13. Show that if $E \subseteq \mathbb{R}^d$, then there exists a G_δ-set $H \supseteq E$ such that $|E|_e = |H|$.

Exercise B.14. Let $E \subseteq \mathbb{R}^d$ be given. Show that the following statements are equivalent.

(a) E is Lebesgue measurable.

(b) For every $\varepsilon > 0$, there exists a closed set $F \subseteq E$ such that $|E \backslash F|_e \leq \varepsilon$.

(c) $E = H \backslash Z$ where H is a G_δ-set and $|Z| = 0$.

(d) $E = H \cup Z$ where H is an F_σ-set and $|Z| = 0$.

The following exercise is a nice application of these characterizations to prove the seemingly "obvious" statement that the measure of a Cartesian product of two sets is the product of the measures of those sets. The proof is surprisingly nontrivial, and consists of proceeding through cases (open sets, G_δ sets, zero measure sets, and finally arbitrary measurable sets).

Exercise B.15. Show that if $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ are Lebesgue measurable, then $E \times F \subseteq R^{m+n}$ is Lebesgue measurable, and $|E \times F| = |E|\,|F|$.

Our statement of Lebesgue measurability, as given in Definition B.6, is formulated in terms of the existence of surrounding open sets, and Exercise B.14 likewise interprets measurability in terms of sets with other topological properties. On the other hand, the equivalent formulation of measurability given in the next theorem does not depend on any topological notions. As such, this *Carathéodory Criterion* is the appropriate definition to use to generalize measurability to more abstract settings, as is done in Appendix D.

Theorem B.16 (Carathéodory's Criterion). *A set $E \subseteq \mathbb{R}^d$ is Lebesgue measurable if and only if*

$$\forall\, A \subseteq \mathbb{R}^d, \quad |A|_e \;=\; |A \cap E|_e + |A \backslash E|_e. \tag{B.2}$$

Proof. $\Rightarrow$. Suppose that E is measurable and that A is any subset of $\mathbb{R}^d$. Since $A = (A \cap E) \cup (A \backslash E)$, we have by subadditivity that $|A|_e \leq |A \cap E|_e + |A \backslash E|_e$. By Exercise B.13, we can find a G_δ-set $H \supseteq A$ such that $|H| = |A|_e$. Note that we can write H as the disjoint union $H = (H \cap E) \cup (H \backslash E)$. Since Lebesgue measure is countably additive and H, E are measurable, we therefore have

$$|A|_e \;=\; |H| \;=\; |H \cap E| + |H \backslash E| \;\geq\; |A \cap E|_e + |A \backslash E|_e,$$

where the final inequality follows from monotonicity.

$\Leftarrow$. Suppose that equation (B.2) holds. Assume first that E is bounded, and let $H \supseteq E$ be a G_δ-set such that $|H| = |E|_e$. Then equation (B.2) implies that

$$|E|_e \;=\; |H| \;=\; |H \cap E|_e + |H \backslash E|_e \;=\; |E|_e + |H \backslash E|_e.$$

Since $|E|_e < \infty$, we conclude that $Z = H \backslash E$ has zero exterior measure and hence is measurable. Since $E = H \backslash Z$, it is measurable as well.

Exercise: By considering the sets $E_k = \{x \in E : |x| \leq k\}$, extend to arbitrary sets E that satisfy equation (B.2). $\square$

B.2.3 Almost Everywhere

Notation B.17. A property that holds except possibly on a set of measure zero is said to hold *almost everywhere*, abbreviated a.e.

For example, if C is the classical Cantor middle-thirds set, then $|C| = 0$ (Problem B.1). Hence, the characteristic function χ_C of C satisfies $\chi_C(x) = 0$ except for those x that belong to the zero measure set C. Therefore we say that $\chi_C(x) = 0$ for almost every x, or $\chi_C = 0$ a.e. for short.

The essential supremum of a function is an example of a quantity that is defined in terms of a property that holds almost everywhere.

Definition B.18 (Essential Supremum). The *essential supremum* of a function $f\colon E \to [-\infty, \infty]$ is

$$\operatorname*{ess\,sup}_{x\in E} f(x) \;=\; \inf\{M : f(x) \le M \text{ a.e.}\}. \tag{B.3}$$

We say that f is *essentially bounded* if

$$\operatorname*{ess\,sup}_{x\in E} |f(x)| \;<\; \infty.$$

Exercise B.19. Show that the infimum in equation (B.3) is achieved, i.e., if we set $M = \operatorname{ess\,sup}_{x\in E} f(x)$ then we have $f(x) \le M$ a.e. In particular, if $\operatorname{ess\,sup}_{x\in E} |f(x)| = 0$, then $f = 0$ a.e.

Additional Problems

B.1. Show that every countable set $E \subseteq \mathbb{R}^d$ satisfies $|E|_e = 0$ and hence is Lebesgue measurable. Show that the Cantor middle-thirds set C is an example of an uncountable subset of $\mathbb{R}$ that satisfies $|C|_e = 0$.

B.2. Define the *inner Lebesgue measure* of a set $E \subseteq \mathbb{R}^d$ to be

$$|E|_i \;=\; \sup\bigl\{|F| : F \text{ closed}, F \subseteq E\bigr\}.$$

Show that if $|E|_e < \infty$, then E is Lebesgue measurable if and only if $|E|_e = |E|_i$, but this can fail if $|E|_e = \infty$.

B.3. Show that continuity from below holds for Lebesgue *exterior* measure, i.e., if $E_1 \subseteq E_2 \subseteq \cdots$ is *any* nested increasing sequence of subsets of $\mathbb{R}^d$, then $|\cup E_k|_e = \lim_{k\to\infty} |E_k|_e$.

B.3 Measurable Functions

One of the goals of this appendix is to define the Lebesgue integral of functions on $\mathbb{R}^d$. We will not be able to integrate every function, and, in particular, the functions that we will integrate will need to be *measurable*. We first define measurability for extended real-valued functions; a complex-valued function will be measurable if its real and imaginary parts are measurable.

Definition B.20 (Real-Valued Measurable Functions). Let $E \subseteq \mathbb{R}^d$ and $f\colon E \to [-\infty, \infty]$ be given. Then f is a *Lebesgue measurable function*, or simply a *measurable function*, if $f^{-1}(\alpha, \infty] = \{x \in E : f(x) > \alpha\}$ is a measurable subset of $\mathbb{R}^d$ for each $\alpha \in \mathbb{R}$.

Notation B.21. For convenience of notation, we often use the abbreviation

$$\{f > a\} \;=\; \{x \in E : f(x) > \alpha\} \;=\; f^{-1}(\alpha, \infty],$$

and other related abbreviations such as $\{f \le g\} = \{x \in E : f(x) \le g(x)\}$.

In particular, every continuous function $f\colon \mathbb{R}^d \to \mathbb{R}$ is measurable. However, a measurable function need not be continuous.

Exercise B.22. Let $E \subseteq \mathbb{R}^d$ be given. Show that E is a Lebesgue measurable set if and only if χ_E is a Lebesgue measurable function.

From now on, we will always implicitly assume that the domain E of a measurable function f is a Lebesgue measurable set. Often, given a particular measurable set E, it will be convenient to consider functions on the domain $E\backslash Z$ where Z is a set of measure zero. In this case we say that f *is defined almost everywhere on* E.

Measurability is preserved under most of the usual operations, including addition, multiplication, and limits. Some care does need to be taken with compositions, but if we compose a measurable function with a continuous function in the correct order, then measurability will be assured.

Exercise B.23. Let $E \subseteq \mathbb{R}^d$ be Lebesgue measurable.

(a) If $f\colon E \to [-\infty, \infty]$ is measurable and $g = f$ a.e., then g is measurable.

(b) If $f,\, g\colon E \to [-\infty, \infty]$ are measurable and finite a.e., then so is $f + g$.

(c) If $f\colon E \to [-\infty, \infty]$ is measurable and finite a.e. and $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous, then $\varphi \circ f$ is measurable. Consequently, $|f|$, f^2, f^+, f^-, and $|f|^p$ for $p > 0$ are all measurable.

(d) If $f,\, g\colon E \to [-\infty, \infty]$ are measurable and finite a.e., then so is fg.

(e) If $f_n\colon E \to [-\infty, \infty]$ are measurable for $n \in \mathbb{N}$, then so are $\sup f_n$, $\inf f_n$, $\limsup f_n$, and $\liminf f_n$.

(f) If $f_n\colon E \to [-\infty, \infty]$ are measurable for $n \in \mathbb{N}$ and $f(x) = \lim_{n\to\infty} f_n(x)$ exists for a.e. x, then f is measurable.

Definition B.24 (Complex-Valued Measurable Functions). Given a domain $E \subseteq \mathbb{R}^d$ and a complex-valued function $f\colon E \to \mathbb{C}$, write f in real and imaginary parts as $f = f_r + i f_i$. Then we say that f is *measurable* if both f_r and f_i are measurable.

Although we will not prove it, Lusin's Theorem states that if f is a measurable function on a compact interval, then we can find a continuous function that equals f except on a set of arbitrarily small measure.

Theorem B.25 (Lusin's Theorem). *If $f\colon [a,b] \to \mathbb{C}$ is Lebesgue measurable and $\varepsilon > 0$, then there exists a continuous function $g\colon [a,b] \to \mathbb{C}$ such that $|\{f \neq g\}| < \varepsilon$.*

Additional Problems

B.4. (a) Show that $f\colon \mathbb{R}^d \to \mathbb{R}$ is Lebesgue measurable if and only if $f^{-1}(U)$ is measurable for every open $U \subseteq \mathbb{R}$.

(b) Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be Lebesgue measurable. Suppose that $g\colon \mathbb{R} \to \mathbb{R}$ is *Borel measurable*, i.e., $g^{-1}(U)$ is a Borel set for every open $U \subseteq \mathbb{R}$ (see Definition D.3). Show that the composition $g \circ f$ is Lebesgue measurable. Also generalize to the case of complex-valued f, g.

B.5. Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous a.e., i.e., f is continuous at almost every point, then f is Lebesgue measurable. Give an example of a function f that is continuous a.e., but such that there is no continuous function g satisfying $f = g$ a.e.

B.6. Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at almost every point, then f' is Lebesgue measurable.

B.4 Convergence in Measure

In this section we consider a particular notion of convergence related to measure.

Definition B.26 (Convergence in Measure). Let $E \subseteq \mathbb{R}^d$ be Lebesgue measurable, and let f_n, f be measurable functions on E that are either complex-valued or are extended real-valued but finite a.e. Then we say that f_n *converges in measure to f on E*, and write $f_n \stackrel{\mathrm{m}}{\to} f$, if

$$\forall\, \varepsilon > 0, \quad \lim_{n\to\infty} \bigl|\{x \in E : |f(x) - f_n(x)| > \varepsilon\}\bigr| \;=\; 0.$$

In general, convergence in measure does not imply pointwise convergence.

Exercise B.27. Give an example of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ such that $f_n \stackrel{\mathrm{m}}{\to} 0$, but $f_n(x)$ does not converge pointwise to zero as $n \to \infty$.

On the other hand, a sequence of functions that converges in measure will always have a subsequence that converges pointwise almost everywhere.

Exercise B.28. Let $E \subseteq \mathbb{R}^d$ be Lebesgue measurable. Assume either that f_n, $f\colon E \to [-\infty, \infty]$ are measurable and finite a.e., or that f_n, $f\colon E \to \mathbb{C}$ are measurable. If $f_n \xrightarrow{\mathrm{m}} f$, show there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ pointwise a.e.

There is also a Cauchy criterion for convergence in measure.

Exercise B.29. Let $E \subseteq \mathbb{R}^d$ be Lebesgue measurable. Assume either that f_n, $f\colon E \to [-\infty, \infty]$ are measurable and finite a.e., or that f_n, $f\colon E \to \mathbb{C}$ are measurable, and show that the following statements are equivalent.

(a) There exists a measurable f such that $f_n \xrightarrow{\mathrm{m}} f$.

(b) For each $\varepsilon > 0$ there exists an $N > 0$ such that $\bigl|\{|f_m - f_n| > \varepsilon\}\bigr| < \varepsilon$ for all m, $n > N$.

B.5 The Lebesgue Integral

To define the Lebesgue integral of a measurable function, we first begin with "simple functions" and then extend through cases to nonnegative functions, real-valued functions, and complex-valued functions.

B.5.1 Integration of Nonnegative Simple Functions

Definition B.30 (Simple Functions). Let $E \subseteq \mathbb{R}^d$ be Lebesgue measurable. A *simple function* on E is a function $\phi\colon E \to \mathbb{C}$ of the form

$$\phi = \sum_{k=1}^{N} a_k \, \chi_{E_k}, \tag{B.4}$$

where $N > 0$, $a_k \in \mathbb{C}$, and the E_k are Lebesgue measurable subsets of E.

Thus, a simple function is a measurable function that takes only finitely many distinct scalar values. If $a_1, \dots, a_N \in \mathbb{C}$ are the distinct values assumed by a simple function ϕ and we set $E_k = \{\phi = a_k\}$, then ϕ has the form given in equation (B.4) and furthermore the sets $E_1, \dots, E_N$ form a partition of E. We call this the *standard representation* of ϕ.

Note that if ϕ and ψ are simple functions on E, then so are $\phi + \psi$ and $\phi\,\psi$.

The integral of a nonnegative simple function is the measure of the region under its graph (recall our convention that $0 \cdot \infty = 0$).

Definition B.31. If ϕ is a nonnegative simple function on E with standard representation $\phi = \sum_{k=1}^{N} a_k \, \chi_{E_k}$, then the *Lebesgue integral of ϕ over E* is

$$\int \phi \;=\; \int \phi(x)\, dx \;=\; \sum_{k=1}^{N} a_k \, |E_k|.$$

Note that if ϕ is a simple function on E and $A \subseteq E$ is measurable, then $\phi\chi_A$ is a simple function on A, and $\int_A \phi = \int_E \phi\, \chi_A$.

Exercise B.32. Prove the following facts for nonnegative simple functions ϕ, ψ on $E \subseteq \mathbb{R}^d$.

(a) $\int_E c\phi = c \int_E \phi$ for $c \geq 0$.

(b) $\int_E (\phi + \psi) = \int_E \phi + \int_E \psi$.

(c) If $\phi \leq \psi$ then $\int_E \phi \leq \int_E \psi$.

(d) $\int_E \phi = 0$ if and only if $\phi = 0$ a.e.

(e) If $A_1, A_2, \dots$ are disjoint measurable subsets of E and $A = \cup A_k$, then $\int_A \phi = \sum_{k=1}^{\infty} \int_{A_k} \phi$.

Remark B.33. In the language of Appendix D, the countable additivity property given in part (e) of Exercise B.32 implies that $\mu(A) = \int_A \phi$ defines a positive measure on the Lebesgue measurable subsets of E. Just as Exercise B.10 shows that Lebesgue measure is continuous from below, the same is true for the measure μ. That is, if $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \cup A_k$, then

$$\int_A \phi \;=\; \lim_{k\to\infty} \int_{A_k} \phi. \tag{B.5}$$

B.5.2 Integration of Nonnegative Functions

We obtain the integral of a nonnegative function by considering all possible approximations from below by simple functions.

Definition B.34 (Lebesgue Integral of a Nonnegative Function). Let $E \subseteq \mathbb{R}^d$ be Lebesgue measurable. If $f\colon E \to [0,\infty]$ is a measurable function, then the *Lebesgue integral of f over E* is

$$\int_E f \;=\; \int_E f(x)\, dx \;=\; \sup\biggl\{\int_E \phi : 0 \leq \phi \leq f,\ \phi \text{ simple}\biggr\}.$$

If A is a measurable subset of E, then we write $\int_A f$ to mean $\int_E f\, \chi_A$. When $E = \mathbb{R}^d$, we write simply $\int f$ or $\int f(x)\, dx$ to denote the integral of f over $\mathbb{R}^d$.

Following are some of the basic properties of integrals of nonnegative functions.

Exercise B.35. Let $E \subseteq \mathbb{R}^d$ and $f, g\colon E \to [0,\infty]$ be measurable, and prove the following facts.

(a) If ϕ is a simple function on E, then the integrals of ϕ given in Definitions B.31 and B.34 coincide.

(b) If $f \le g$ then $\int_E f \le \int_E g$.

(c) Tchebyshev's Inequality: If $\alpha > 0$, then $|\{x \in E : f(x) > \alpha\}| \le \frac{1}{\alpha}\int_E f$.

(d) $\int_E f = 0$ if and only if $f = 0$ a.e.

The definition of $\int_E f$ given in Definition B.34 is often cumbersome to implement. One application of the next result (which is also known as the *Beppo-Levi Theorem*) is that we will be able to obtain the integral of f as a *limit* instead of supremum of integrals of simple functions. This is quite useful, since limits are linear while suprema are not in general.

We say that a sequence of extended real-valued functions $\{f_n\}_{n\in\mathbb{N}}$ is *monotone increasing* if $f_1(x) \le f_2(x) \le \cdots$ for all x. We write $f_n \nearrow f$ to mean that $\{f_n\}_{n\in\mathbb{N}}$ is monotone increasing and $f_n(x) \to f(x)$ pointwise.

Theorem B.36 (Monotone Convergence Theorem). *Let $E \subseteq \mathbb{R}^d$ be measurable, and assume $\{f_n\}_{n\in\mathbb{N}}$ are nonnegative measurable functions on E such that $f_n \nearrow f$. Then*

$$\lim_{n\to\infty} \int_E f_n \;=\; \int_E f.$$

Proof. Since $\left\{\int_E f_n\right\}_{n\in\mathbb{N}}$ is an increasing sequence of real scalars, $I = \lim_{n\to\infty}\int_E f_n$ exists as an extended real number. Further, $0 \le I \le \int_E f$ since $0 \le f_n \le f$ for all n.

Choose any simple function ϕ with $0 \le \phi \le f$, and fix $0 < \alpha < 1$. Set $E_n = \{f_n \ge \alpha\phi\}$ and observe that $E_1 \subseteq E_2 \subseteq \cdots$ and $\cup E_n = E$. By the continuity from below property given in equation (B.5), we therefore have that $\int_{E_n}\phi \to \int_E \phi$. Consequently,

$$\int_E f_n \;\ge\; \int_{E_n} f_n \;\ge\; \int_{E_n} \alpha\phi \;\to\; \alpha\int_E \phi \quad \text{as } n \to \infty,$$

and therefore $I \ge \alpha\int_E \phi$. Since $\alpha < 1$ is arbitrary it follows that $I \ge \int_E \phi$. Taking the supremum over all such simple functions ϕ, we conclude that $I \ge \int_E f$. ◻

As an application of the Monotone Convergence Theorem, we can prove additivity of the integral of nonnegative functions. In order to do this, we need to show that we can always find simple functions that increase pointwise to an arbitrary nonnegative measurable function.

Exercise B.37. Suppose that $E \subseteq \mathbb{R}^d$ and $f, g\colon E \to [0,\infty]$ are measurable.

(a) Show that

$$\phi_n(x) = \begin{cases} \dfrac{j-1}{2^n}, & \text{if } \dfrac{j-1}{2^n} \le f(x) < \dfrac{j}{2^n},\ j = 1, \dots, n2^n, \\ n, & \text{if } f(x) \ge n. \end{cases} \qquad \text{(B.6)}$$

is a simple function, $\phi_n(x) \nearrow f(x)$ for each x, and if f is bounded then ϕ_n converges uniformly to f. Apply the Monotone Convergence Theorem to conclude that $\int_E \phi_n \nearrow \int_E f$.

(b) Show that

$$\int_E (f+g) = \int_E f + \int_E g.$$

Since the partial sums of a series of nonnegative functions form a monotone increasing sequence, we can apply the Monotone Convergence Theorem to the issue of interchanging a sum with an integral.

Corollary B.38. *If $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of measurable, nonnegative functions on a measurable set $E \subseteq \mathbb{R}^d$, then*

$$\int_E \biggl(\sum_{n=1}^{\infty} f_n\biggr) = \sum_{n=1}^{\infty} \int_E f_n.$$

In particular, if $f\colon E \to [0, \infty]$ is measurable, $A_1, A_2, \dots$ are disjoint and measurable, and $A = \cup A_k$, then

$$\int_A f = \sum_k \int_{A_k} f.$$

If we have functions f_n that are not monotone increasing, then we may not be able to interchange a limit with an integral. On the other hand, the following result states that as long as the f_n are all nonnegative, we do at least have an inequality.

Exercise B.39 (Fatou's Lemma). If $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of measurable, nonnegative functions on a measurable set $E \subseteq \mathbb{R}^d$, then

$$\int_E \Bigl(\liminf_{n\to\infty} f_n\Bigr) \le \liminf_{n\to\infty} \int_E f_n.$$

Exercise B.40. Show that strict inequality can hold in Fatou's Lemma.

Since changing the value of a function on a set of zero measure does not change the value of its integral, it suffices to assume that the hypotheses in the Monotone Convergence Theorem hold a.e. instead of everywhere, and similarly for other theorems in this section.

B.5.3 Integration of Real-Valued and Complex-Valued Functions

We define the integral of a general real-valued function by writing it as a difference of two nonnegative functions.

Definition B.41 (Lebesgue Integral of a Real-Valued Function). Let $E \subseteq \mathbb{R}^d$ be measurable, and suppose that a measurable extended real-valued function $f\colon E \to [-\infty, \infty]$ is given. Define

$$f^+(x) \;=\; \max\bigl\{f(x),\, 0\bigr\}, \qquad f^-(x) \;=\; \max\bigl\{-f(x),\, 0\bigr\}.$$

Then $f^+,\ f^- \geq 0$, and we have $f = f^+ - f^-$ and $|f| = f^+ + f^-$. We define the *Lebesgue integral of f on E* to be

$$\int_E f \;=\; \int_E f^+ \;-\; \int_E f^-,$$

as long as this does not have the form $\infty - \infty$ (in that case, the integral is undefined).

Similarly, the integral of a complex-valued function is defined by breaking into real and imaginary parts.

Definition B.42 (Lebesgue Integral of a Complex-Valued Function). Let $E \subseteq \mathbb{R}^d$ be measurable, and let $f\colon E \to \mathbb{C}$ be a measurable complex-valued function on E. Write the real and imaginary parts of f as $f = f_r + i f_i$. If $\int_E f_r$ and $\int_E f_i$ both exist and are finite, then the *Lebesgue integral of f on E* is

$$\int_E f \;=\; \int_E f_r \;+\; i \int_E f_i.$$

Note that if f is complex-valued and $\int_E f$ exists then, by definition, $\int_E f$ is a (finite) complex scalar.

Exercise B.43. Given an extended real-valued or a complex-valued measurable function f, show that $\int_E f$ exists and is a finite scalar if and only if $\int_E |f| < \infty$. Further, in this case we have $\bigl|\int_E f\,\bigr| \leq \int_E |f|$.

B.5.4 The Lebesgue Dominated Convergence Theorem

We have already seen several theorems, including the Monotone Convergence Theorem and Fatou's Lemma, that deal with the issue of interchanging a limit and an integral. The Dominated Convergence Theorem is the workhorse of the stable of convergence theorems.

Exercise B.44 (Lebesgue Dominated Convergence Theorem). Assume $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions on a measurable set $E \subseteq \mathbb{R}^d$, either extended real-valued or complex-valued, such that:

(a) $f(x) = \lim_{n\to\infty} f_n(x)$ exists for a.e. $x \in E$, and
(b) there exists $g \in L^1(E)$ such that $|f_n(x)| \le g(x)$ a.e. for every n.

Then f_n converges to f in L^1-norm, i.e.,

$$\lim_{n\to\infty} \|f - f_n\|_1 \;=\; \lim_{n\to\infty} \int_E |f - f_n| \;=\; 0,$$

and, consequently,

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

B.5.5 Relation to the Riemann Integral

For functions defined on finite intervals $[a, b]$, we can ask how the Lebesgue integral of f relates to its Riemann integral. Unfortunately, not every Lebesgue integrable function is Riemann integrable. For example, the characteristic function $\chi_{\mathbb{Q}}$ of the rational numbers is Lebesgue integrable but not Riemann integrable. We state without proof the following characterization of Riemann integrability.

Theorem B.45. *A bounded function f is Riemann integrable on $[a, b]$ if and only if f is continuous at almost every point in $[a, b]$.*

Note that continuity a.e. does not imply that there exists a continuous function g such that $f = g$ a.e.

Although not every Lebesgue integrable function is Riemann integrable, the good news is that, on a finite interval, every Riemann integrable function is Lebesgue integrable, and the two integrals agree.

Theorem B.46. *If f is a bounded function that is Riemann integrable on a finite interval $[a, b]$, then it is Lebesgue integrable on $[a, b]$ and its Riemann integral equals its Lebesgue integral $\int_a^b f$.*

The situation is somewhat more complicated when dealing with improper Riemann integrals. For example, Problem B.18 below shows that the improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\, dx \;=\; \lim_{a\to\infty} \int_0^a \frac{\sin x}{x}\, dx \;=\; \frac{\pi}{2}$$

exists. However, if we set $f(x) = \frac{\sin x}{x}$ then we have both $\int_0^\infty f^+ = \infty$ and $\int_0^\infty f^- = \infty$, so the Lebesgue integral of f on $[0, \infty)$ does not exist. In essence, improper Riemann integrals may exist because of fortunate cancellations, while the Lebesgue integral requires "absolute convergence."

Additional Problems

B.7. Let $f\colon \mathbb{R} \to \mathbb{C}$ be Lebesgue measurable. Show that there exist simple functions ϕ_k such that $\phi_k(x) \to f(x)$ pointwise as $k \to \infty$, $|\phi_k(x)| \le |f(x)|$ for every k and x, and the convergence is uniform on every set on which f is bounded.

B.6 The L^p Spaces

In this section we introduce and examine the Lebesgue spaces $L^p(E)$, which are Banach spaces for $1 \le p \le \infty$ and complete metric spaces for $0 < p < 1$.

B.6.1 Norm and Completeness

Definition B.47 (Integrable Function). Let $E \subseteq \mathbb{R}^d$ be measurable. Then a measurable function f on E (either extended real-valued or complex-valued) is *integrable* on E if $\int_E |f| < \infty$.

Note that any integrable function must be finite almost everywhere.

Definition B.48. Let $E \subseteq \mathbb{R}^d$ be measurable.

(a) If $0 < p < \infty$, then $L^p(E)$ consists of all measurable functions $f\colon E \to \mathbb{C}$ such that $|f|^p$ is integrable, i.e.,

$$\|f\|_p \;=\; \left(\int_E |f|^p\right)^{1/p} \;<\; \infty.$$

(b) For $p = \infty$, the space $L^\infty(E)$ consists of all those measurable functions $f\colon E \to \mathbb{C}$ for which $|f|$ is essentially bounded, i.e.,

$$\|f\|_\infty \;=\; \operatorname*{ess\,sup}_{x\in E} |f(x)| \;<\; \infty.$$

We refer to $L^p(\mathbb{R})$ for $p < \infty$ as the *Lebesgue space of p-integrable functions*, and to $L^\infty(\mathbb{R})$ as the *Lebesgue space of essentially bounded functions*.

The proof of Hölder's Inequality for ℓ^p, given in Theorem A.16, carries over to $L^p(E)$.

Exercise B.49 (Hölder's Inequality). Let $E \subseteq \mathbb{R}$ be measurable, and fix $1 \le p \le \infty$. Let p' denote the dual index to p (see Section A.1). If $f \in L^p(E)$ and $g \in L^{p'}(E)$ then $fg \in L^1(E)$, and

$$\|fg\|_1 \le \|f\|_p \, \|g\|_{p'}.$$

For $1 < p < \infty$, this inequality is

$$\int_E |fg| \;\le\; \left(\int_E |f|^p\right)^{1/p} \left(\int_E |g|^{p'}\right)^{1/p'}.$$

Equality holds in Hölder's Inequality if and only if there exist scalars α, β, not both zero, such that $\alpha\,|f(x)|^p = \beta\,|g(x)|^{p'}$ a.e. (Problem B.12).

We saw in Exercise A.17 that, as a consequence of Hölder's Inequality, the ℓ^p spaces are Banach spaces for $p \geq 1$. Unfortunately, for the Lebesgue spaces, $\int_E |f|^p = 0$ only implies that $f = 0$ a.e. Consequently, while $\|\cdot\|_p$ is a seminorm, it is not a norm on $L^p(E)$.

Exercise B.50. If $E \subseteq \mathbb{R}^d$ is measurable and $1 \leq p \leq \infty$, then $\|\cdot\|_p$ is a seminorm on $L^p(E)$.

The Triangle Inequality on L^p is also known as *Minkowski's Inequality.*

By making some appropriate identifications, we can deal with the technical complication that $\|\cdot\|_p$ is only a seminorm, and in the end prove Banach space properties of the L^p spaces. The basic problem is illustrated by the fact that if E is any set of measure zero then $\|\chi_E\|_p = 0$ even though χ_E is not the zero function. However, we do have $\chi_E = 0$ a.e., which suggests that when dealing with the L^p spaces we may not wish to distinguish between functions that are equal almost everywhere. Indeed, the next exercise is the standard procedure for "converting" a seminorm into a norm by forming equivalence classes.

Exercise B.51. Show that the relation $f \sim g$ if $f = g$ a.e. is an equivalence relation on $L^p(E)$. Let $\widetilde{f}$ denote the equivalence class of f in $L^p(E)$ under this relation, and set $|||\widetilde{f}|||_p = \|f\|_p$. Show that this quantity is independent of the choice of representative f of $\widetilde{f}$. Let the quotient space $\widetilde{L^p}(E)$ consist of all the distinct equivalence classes of $f \in L^p(E)$, i.e.,

$$\widetilde{L^p}(E) \;=\; \bigl\{\widetilde{f} : f \in L^p(E)\bigr\},$$

and show that $\widetilde{L^p}(E)$ is a normed space with respect to $|||\cdot|||_p$.

Typically we abuse notation and let the symbol f denote the equivalence class $\widetilde{f}$ of all functions equal to f a.e., and we write $L^p(E)$ instead of $\widetilde{L^p}(E)$. In other words, we identify any two functions that are equal a.e. Adopting this convention, we will shortly see that the L^p spaces are not only normed spaces, but are complete.

B.6.2 On Abuses of Notation

Ignoring the distinction between a function and the equivalence class of functions that are equal to it a.e. is not usually a problem, but on occasion some care needs to be taken. One such situation arises when dealing with continuous functions. Every function in $C_b(\mathbb{R})$ is continuous and bounded, so we often write $C_b(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$. However, in doing so we are really identifying $C_b(\mathbb{R})$ with its image in $L^\infty(\mathbb{R})$ under the equivalence relation $\sim$, i.e., if $f \in C_b(\mathbb{R})$ then it determines an equivalence class $\widetilde{f}$ of functions that are equal to it almost everywhere, and it is this equivalence class $\widetilde{f}$ that belongs to $L^\infty(\mathbb{R})$. Conversely, if we are given $f \in L^\infty(\mathbb{R})$ (really an equivalence class $\widetilde{f}$ of functions) and there is a representative of this equivalence class that belongs to $C_b(\mathbb{R})$, then we write $f \in C_b(\mathbb{R})$, meaning that there is a representative of f that belongs to $C_b(\mathbb{R})$.

Remark B.52. The two statements "f is continuous a.e." and "f equals a continuous function a.e." are distinct. The first means that $\lim_{y\to x} f(y) = f(x)$ for almost every x, while the second means that there exists a continuous function g such that $f(x) = g(x)$ for almost every x. Only in the latter case can we say that there is a representative of f that is a continuous function.

Exercise B.53. Show that if $f \in C_b(\mathbb{R})$, then

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| \;=\; \sup_{x\in\mathbb{R}} |f(x)|.$$

Consequently, for continuous bounded functions, the uniform norm defined in Exercise A.20 coincides with the L^∞-norm defined above.

Another place where the fact that elements of $L^p(E)$ are equivalence classes must be taken into account is when discussing the support of a function in $L^p(E)$. For example, $\chi_{\mathbb{Q}}$ is one representative of the zero function in $L^p(\mathbb{R})$, yet the support of $\chi_{\mathbb{Q}}$ is the entire real line. Unfortunately, the support of a function depends very much on the choice of representative. Still, it is such a convenient concept that we usually abuse notation and apply support terminology to elements of $L^p(E)$. For example, we write "f has compact support" with the understanding that this means that some representative of f has compact support. Thus, f has compact support if $f(x) = 0$ for almost every x outside of some compact set K.

When dealing with support issues, it is often enough to know that a function is zero almost everywhere outside of some particular set. However, it is sometimes necessary to be more precise about the support of a function. In this volume, we adopt the following definition of the support of $f \in L^p(\mathbb{R}^d)$. This definition corresponds to taking the support of f to be the support of the distribution determined by f, see Section 4.5. This definition can be adapted to elements of $L^p(E)$, although some care should be taken if E is not a closed subset of $\mathbb{R}^d$.

Definition B.54. The *support* of $f \in L^p(\mathbb{R}^d)$ is

$$\operatorname{supp}(f) \;=\; \bigcap \bigl\{F \subseteq \mathbb{R}^d : F \text{ is closed and } f(x) = 0 \text{ for a.e. } x \notin F\bigr\}.$$

In particular, if $F \subseteq \mathbb{R}^d$ is closed, then

$$\operatorname{supp}(f) \subseteq F \iff f(x) = 0 \text{ for a.e. } x \notin F.$$

Although we now have two definitions of support (Definition A.19 and Definition B.54), it is usually clear from context which is meant. If a given function f is defined everywhere then we apply Definition A.19, while if f represents an equivalence class of functions that are equal almost everywhere, then we use Definition B.54. If f is a continuous function (in the sense that there is a representative of f that is continuous), then the two definitions coincide.

B.6.3 Convergence in $L^p(E)$

In the ℓ^p spaces, convergence in ℓ^p norm implies componentwise convergence (see Problem A.6). The situation in $L^p(E)$ is a little different. For $p = \infty$, if $f_n \to f$ in $L^\infty(E)$ then it follows that $f_n(x) \to f(x)$ pointwise a.e. However, for p finite, an L^p-convergent sequence need not converge pointwise.

Exercise B.55. Let $0 < p < \infty$ be fixed. Give an example of functions $f_n \in L^p(\mathbb{R})$ such that $f_n \to 0$ in L^p-norm (i.e., $\|f_n\|_p \to 0$), but $f_n(x)$ does not converge pointwise to zero a.e. as $n \to \infty$.

Fortunately, it is true that an L^p-convergent sequence always has a subsequence that converges pointwise almost everywhere.

Exercise B.56. Let $E \subseteq \mathbb{R}^d$ be measurable and fix $0 < p \le \infty$. Show that if $f_n, f \in L^p(E)$ and $f_n \to f$ in $L^p(E)$, then $f_n \xrightarrow{\mathrm{m}} f$. Consequently, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k}(x) \to f(x)$ for almost every $x \in E$.

The preceding exercise can be used to prove the completeness of the L^p spaces.

Exercise B.57. Let $E \subseteq \mathbb{R}^d$ be measurable, and prove the following statements.

(a) If $1 \le p \le \infty$, then $\|\cdot\|_p$ is a norm on $L^p(E)$, and $L^p(E)$ is a Banach space with respect to this norm.

(b) If $0 < p < 1$, then $\mathrm{d}(f,g) = \|f-g\|_p^p$ is a metric on $L^p(E)$, and $L^p(E)$ is complete with respect to this metric.

The space $L^2(E)$ is special. As before, we consider elements of $L^2(E)$ to be equivalence classes of functions that are equal almost everywhere.

Exercise B.58. If $E \subseteq \mathbb{R}^d$ is measurable, show that $\langle f, g\rangle = \int_E f(x)\,\overline{g(x)}\,dx$ defines an inner product on $L^2(E)$, and $L^2(E)$ is a Hilbert space with respect to this inner product.

It is often useful to know that we can approximate a given L^p function by functions that have some special properties. For example, combining the Lebesgue Dominated Convergence Theorem with Exercise B.37, we see that the L^p simple functions are dense in $L^p(E)$. In fact, we can even restrict further to simple functions with compact support when p is finite.

Exercise B.59. Let $E \subseteq \mathbb{R}^d$ be Lebesgue measurable. Show that the set of all simple functions on E is dense in $L^p(E)$ for each $1 \le p \le \infty$, and the set of compactly supported simple functions is dense in $L^p(E)$ for $1 \le p < \infty$.

We can then use the denseness of the simple functions to prove that the space of continuous, compactly supported functions is dense in $L^p(\mathbb{R}^d)$ for finite p. For $p = \infty$, we have instead that $C_c(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$ in the uniform norm.

Theorem B.60. *$C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for each $1 \le p < \infty$.*

Proof. First consider the function $f = \chi_E$ where $E \subseteq \mathbb{R}^d$ is bounded. If we fix $\varepsilon > 0$, then there exists a bounded open set $U \supseteq E$ such that $|U \backslash E| < \varepsilon$. By Problem B.2, we can also find a compact set $K \subseteq E$ such that $|E \backslash K| < \varepsilon$. By Urysohn's Lemma (Theorem A.109), we can find a continuous function $\theta \colon \mathbb{R}^d \to \mathbb{R}$ such that $0 \le \theta \le 1$, $\theta = 1$ on K, and $\theta = 0$ on $\mathbb{R}^d \backslash U$. Then $\theta \in C_c(\mathbb{R}^d)$, and we have

$$\|\chi_E - \theta\|_p^p \;=\; \int |\chi_E - \theta|^p \;=\; \int_{U \backslash K} |\chi_E - \theta|^p \;\le\; |U \backslash K| \;<\; 2\varepsilon.$$

Hence χ_E can be approximated arbitrarily closely in L^p-norm by elements of $C_c(\mathbb{R}^d)$. Exercise: Complete the proof by making use of Exercise B.59. □

For $p = \infty$, the space $C_c(\mathbb{R}^d)$ is not dense in $L^\infty(\mathbb{R}^d)$, but it is dense in $C_0(\mathbb{R}^d)$ with respect to the uniform norm.

Theorem B.60 can be used to show that the space of finite linear combinations of characteristic functions of intervals (sometimes called "really simple functions") is dense in $L^p(\mathbb{R})$. This provides us with another useful set of approximating functions for $L^p(\mathbb{R})$.

Exercise B.61. Show that $\{\chi_{[a,b]} : -\infty < a < b < \infty\}$ is complete in $L^p(\mathbb{R})$ when $1 \le p < \infty$.

B.6.4 Local Integrability

We introduce one final space in this section. This space is not a Banach space, but it is very useful when we only need to consider integrability at a "local" level.

Definition B.62 (Locally Integrable Functions). The space of *locally integrable functions* on $\mathbb{R}^d$ is

$$L^1_{\mathrm{loc}}(\mathbb{R}^d) \;=\; \bigl\{f \colon \mathbb{R}^d \to \mathbb{C} : f \cdot \chi_K \in L^1(\mathbb{R}^d) \text{ for every compact } K \subseteq \mathbb{R}^d\bigr\}.$$

As with the L^p spaces, we regard the elements of $L^1_{\rm loc}(\mathbb{R}^d)$ as being equivalence classes of functions that are equal almost everywhere. $L^1_{\rm loc}(\mathbb{R}^d)$ is not a Banach space, but it is a typical example of a topological vector space whose topology is defined by an infinite family of seminorms instead of a single norm (see Example E.5).

If we likewise define $L^p_{\rm loc}(\mathbb{R}^d)$ for $p > 1$, then we have

$$L^p(\mathbb{R}^d) \subsetneq L^p_{\rm loc}(\mathbb{R}^d) \subsetneq L^1_{\rm loc}(\mathbb{R}^d), \qquad 1 \le p \le \infty.$$

$L^1_{\rm loc}(\mathbb{R}^d)$ contains many functions that do not belong to any $L^p(\mathbb{R}^d)$. For example, every polynomial belongs to $L^1_{\rm loc}(\mathbb{R})$, as does e^x.

Additional Problems

B.8. Let $f \in L^1(\mathbb{R}^d)$ be given. Given $\varepsilon > 0$, show that there exists a $\delta > 0$ such that for any measurable $E \subseteq \mathbb{R}^d$ satisfying $|E| < \delta$, we have $\int_E |f| < \varepsilon$. In particular, if $|E| = 0$, then $\int_E f = 0$.

B.9. Show that if $|E| < \infty$ and $0 < p \le q \le \infty$, then $L^q(E) \subseteq L^p(E)$. In contrast, show that $L^p(\mathbb{R})$ is not contained in $L^q(\mathbb{R})$ for any p and q.

B.10. Prove that if $|E| < \infty$, then $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

B.11. Show that $L^p(\mathbb{R})$ is separable for $1 \le p < \infty$, but $L^\infty(\mathbb{R})$ is not separable.

B.12. Show that equality holds in Hölder's Inequality if and only if there exist scalars α, β, not both zero, such that $\alpha\,|f|^p = \beta\,|g|^{p'}$ a.e.

B.13. This problem generalizes Hölder's Inequality to the case of more than two functions. Show that if $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1$ and $f_i \in L^{p_i}(\mathbb{R})$ for $i = 1, \dots, k$, then $f_1 \cdots f_k \in L^1(\mathbb{R})$ and $\|f_1 \cdots f_k\|_1 \le \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}$.

B.14. Given $1 \le p < q \le \infty$, show that $L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ is a Banach space under the norm $\|f\| = \|f\|_p + \|f\|_q$. Further, if $1 \le p < r < q \le \infty$ then we have $L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subseteq L^r(\mathbb{R})$, with

$$\|f\|_r \le \|f\|_p^\theta\, \|f\|_q^{1-\theta} \quad \text{where } \frac{1-\theta}{p} + \frac{\theta}{q} = \frac{1}{r}.$$

Show also that if $r < \infty$ then $L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ is dense in $L^r(\mathbb{R})$.

B.15. Given $1 \le p < q \le \infty$, show that

$$L^p(\mathbb{R}) + L^q(\mathbb{R}) = \bigl\{f + g : f \in L^p(\mathbb{R}), g \in L^q(\mathbb{R})\bigr\}$$

is a Banach space under the norm

$$\|f\| = \inf\bigl\{\|g\|_p + \|h\|_q : f = g + h \text{ with } g \in L^p(\mathbb{R}), h \in L^q(\mathbb{R})\bigr\}.$$

Further, if $1 \le p < r < q \le \infty$ then we have $L^r(\mathbb{R}) \subseteq L^p(\mathbb{R}) + L^q(\mathbb{R})$.

B.7 Repeated Integration

Let $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ be measurable. If f is a measurable function on $E \times F$ then there are three natural integrals of f over $E \times F$. First, there is the integral of f over the set $E \times F \subseteq \mathbb{R}^{m+n}$, which we write as the *double integral*

$$\iint_{E\times F} f \;=\; \iint_{E\times F} f(x,y)\,(dx\,dy).$$

Second, for each fixed y we can integrate $f(x,y)$ as a function of x, and then integrate the result in y, obtaining the *iterated integral*

$$\int_F \left(\int_E f(x,y)\,dx\right) dy.$$

Third, we also have the iterated integral

$$\int_E \left(\int_F f(x,y)\,dy\right) dx.$$

In general these three integrals need not be equal, even if they all exist.

In this section we state without proof the theorems of Fubini and Tonelli, which give sufficient conditions under which we can exchange the order of integration. We begin with Tonelli's Theorem, which states that interchange is allowed if f is nonnegative. In particular, this suggests that a counterexample to equality of the integrals must be related to the indeterminacy of $\infty - \infty$ (see Problem B.16).

Theorem B.63 (Tonelli's Theorem). *Let E be a measurable subset of $\mathbb{R}^m$ and F a measurable subset of $\mathbb{R}^n$. If $f\colon E \times F \to [0,\infty]$ is measurable, then the following statements hold.*

(a) $f_x(y) = f(x,y)$ *is measurable on F for each $x \in E$.*

(b) $f^y(x) = f(x,y)$ *is measurable on E for each $y \in F$.*

(c) $g(x) = \int_F f_x(y)\,dy$ *is a measurable function on E.*

(d) $h(y) = \int_E f^y(x)\,dx$ *is a measurable function on F.*

(e) *As extended real numbers,*

$$\begin{aligned}\iint_{E\times F} f(x,y)\,(dx\,dy) \;&=\; \int_F \left(\int_E f(x,y)\,dx\right) dy \\ &=\; \int_E \left(\int_F f(x,y)\,dy\right) dx. \end{aligned} \tag{B.7}$$

As a corollary, we obtain the useful fact that to test whether a given function belongs to $L^1(E \times F)$ we can simply show that any one of three possible integrals is finite.

Corollary B.64. *Let E be a measurable subset of $\mathbb{R}^m$ and F a measurable subset of $\mathbb{R}^n$. If f is a measurable function on $E \times F$, then (as extended real numbers):*

$$\iint_{E\times F} |f(x,y)|\,(dx\,dy) = \int_F \biggl(\int_E |f(x,y)|\,dx\biggr)\,dy = \int_E \biggl(\int_F |f(x,y)|\,dy\biggr)\,dx.$$

Consequently, if any one of these three integrals is finite, then $f \in L^1(E\times F)$.

Fubini's Theorem allows the interchange of integrals if f is integrable (thereby again avoiding the ambiguity that is $\infty - \infty$).

Theorem B.65 (Fubini's Theorem). *Let E be a measurable subset of $\mathbb{R}^m$ and F a measurable subset of $\mathbb{R}^n$. If $f \in L^1(E \times F)$, then the following statements hold.*

(a) *$f_x(y) = f(x,y)$ is measurable and integrable on F for almost every $x \in E$.*
(b) *$f^y(x) = f(x,y)$ is measurable and integrable on E for almost every $y \in F$.*
(c) *$g(x) = \int_F f_x(y)\,dy$ is a measurable and integrable function on E.*
(d) *$h(y) = \int_E f^y(x)\,dx$ is a measurable and integrable function on F.*
(e) *We have*

$$\iint_{E\times F} f(x,y)(dx\,dy) = \int_F \biggl(\int_E f(x,y)\,dx\biggr)\,dy = \int_E \biggl(\int_F f(x,y)\,dy\biggr)\,dx.$$

Additional Problems

B.16. Show that the following iterated integrals have the indicated values:

$$\begin{aligned}
\int_1^\infty \biggl(\int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\biggr)\,dx &= -\frac{\pi}{4},\\
\int_1^\infty \biggl(\int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\biggr)\,dy &= \frac{\pi}{4},\\
\int_1^\infty \biggl(\int_1^\infty \biggl|\frac{x^2-y^2}{(x^2+y^2)^2}\biggr|\,dx\biggr)\,dy &= \infty.
\end{aligned}$$

Conclude that equality need not hold in equation (B.7) if the hypotheses of Fubini's Theorem are not fulfilled.

B.17. Let $f(x,y)$ be a measurable function on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$, and fix $1 \le p < \infty$. Prove *Minkowski's Integral Inequality*:

$$\biggl(\int \biggl(\int |f(x,y)|\,dy\biggr)^p dx\biggr)^{1/p} \;\le\; \int \biggl(\int |f(x,y)|^p\,dx\biggr)^{1/p} dy. \tag{B.8}$$

Remark: This equation may be more revealing if we rewrite it as

$$\left\| \int |f(\cdot, y)|\, dy \right\|_p \;\le\; \int \|f(\cdot, y)\|_p \, dy.$$

Thus, Minkowski's Integral Inequality is an integral version of the Triangle Inequality (also known as Minkowski's Inequality) on $L^p(\mathbb{R}^m)$.

B.18. If $x > 0$, then $\int_0^\infty e^{-tx}\, dt = \frac{1}{x}$. Combine this with Fubini's Theorem to evaluate the integral $\int_0^a \frac{\sin x}{x}\, dx$. Then apply the Lebesgue Dominated Convergence Theorem to show that $\lim_{a\to\infty} \int_0^a \frac{\sin x}{x}\, dx = \frac{\pi}{2}$. Thus, even though $\frac{sin x}{x}$ is not integrable, the improper Riemann integral $\int_0^\infty \frac{\sin x}{x}\, dx$ exists and equals $\frac{\pi}{2}$ (this integral can also be evaluated by using contour integration).

B.19. This problem will establish a version of *Hardy's Inequalities*.

(a) Given $1 \le p < \infty$ and $\alpha < -1$, show there exists a constant $C(\alpha, p)$ such that for any measurable $f \colon (0,\infty) \to [0,\infty]$ we have

$$\int_0^\infty \left(\int_0^x f(t)\, dt \right)^p x^\alpha \, dx \;\le\; C(\alpha, p) \int_0^\infty f(t)^p\, t^{\alpha+p}\, dt. \tag{B.9}$$

Show that if $\alpha > -1$ then the inequality is

$$\int_0^\infty \left(\int_x^\infty f(t)\, dt \right)^p x^\alpha \, dx \;\le\; C(\alpha, p) \int_0^\infty f(t)^p\, t^{\alpha+p}\, dt.$$

(b) For the case $\alpha = -p < -1$, show that the optimal constant in equation (B.9) is

$$C(-p, p) \;=\; (p')^p \;=\; \left(\frac{p}{p-1} \right)^p.$$

(c) Suppose that $f \in L^p(\mathbb{R})$ where $1 < p < \infty$. Define $F(x) = \frac{1}{x} \int_0^x |f(t)|\, dt$ for $x \in \mathbb{R}$, and show that

$$\|F\|_p \;\le\; p'\, \|f\|_p, \tag{B.10}$$

with p' being the best possible constant. Also show that equality holds in equation (B.10) if and only if $f = 0$ a.e.

B.8 Functions of Bounded Variation

In this section we briefly review the definition and basic properties of functions with bounded variation on the real line.

B.8.1 Definition and Examples

We begin by defining bounded variation for functions on finite closed intervals in $\mathbb{R}$.

Definition B.66. Let $f \colon [a,b] \to \mathbb{C}$ be given. Given any finite partition $\Gamma = \{a = x_0 < \cdots < x_n = b\}$ of $[a,b]$, set

$$S_\Gamma \;=\; \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|.$$

The *variation* of f over $[a,b]$ is

$$V[f;a,b] \;=\; \sup\bigl\{S_\Gamma : \Gamma \text{ is a partition of } [a,b]\bigr\}.$$

The function f has *bounded variation* on $[a,b]$ if $V[f;a,b] < \infty$. We set

$$\mathrm{BV}[a,b] \;=\; \bigl\{f \colon [a,b] \to \mathbb{C} : f \text{ has bounded variation}\bigr\}.$$

Note that in this definition we are considering f to be a function that is defined at all points, rather than an equivalence class of functions that are equal a.e.

The space $\mathrm{BV}[a,b]$ is sometimes defined to consist of real-valued functions of bounded variation, but all the definitions and results extend to complex-valued functions, which is the setting of most of this volume. We could just as well have defined bounded variation for real-valued functions, and then declared a complex-valued function to have bounded variation if its real and imaginary parts have bounded variation.

Exercise B.67. Given $f \colon [a,b] \to \mathbb{C}$, write the real and imaginary parts as $f = f_r + i f_i$. Show that $f \in \mathrm{BV}[a,b]$ if and only if $f_r,\, f_i \in \mathrm{BV}[a,b]$.

For functions on the domain $\mathbb{R}$ we make the following definition.

Definition B.68. The *variation* of a function $f \colon \mathbb{R} \to \mathbb{C}$ is

$$V[f;\mathbb{R}] \;=\; \sup_{a<b} V[f;a,b].$$

We say that f has *bounded variation* if $V[f;\mathbb{R}] < \infty$, and we define

$$\mathrm{BV}(\mathbb{R}) \;=\; \bigl\{f \colon \mathbb{R} \to \mathbb{C} : f \text{ has bounded variation}\bigr\}.$$

Example B.69. A function $f \colon [a,b] \to \mathbb{R}$ is *monotone increasing* if

$$a \le x \le y \le b \implies f(x) \le f(y).$$

In this case f has bounded variation, and $V[f;a,b] = f(b) - f(a)$.

Exercise B.70. Define $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show that f is continuous, but has unbounded variation on $[-1, 1]$.

Lipschitz functions on bounded domains all have bounded variation.

Definition B.71. A function $f\colon [a,b] \to \mathbb{C}$ is *Lipschitz* on $[a,b]$ if there exists a constant $C > 0$ (called a *Lipschitz constant* for f) such that

$$|f(x) - f(y)| \;\le\; C\,|x - y|, \qquad x, y \in [a,b].$$

The class of Lipschitz functions on $[a,b]$ is denoted by

$$\mathrm{Lip}[a,b] \;=\; \bigl\{f\colon [a,b] \to \mathbb{C} : f \text{ is Lipschitz}\bigr\}.$$

Notation B.72. We will often need to discuss the differentiability properties of functions on finite intervals. We will say that a function f is *everywhere differentiable on* $[a,b]$ if it is differentiable on the interior (a,b) and if the appropriate one-sided derivatives exist at the endpoints. In other words, f is everywhere differentiable on $[a,b]$ if

$$\lim_{\substack{y \to x,\\ y \in [a,b]}} \frac{f(y) - f(x)}{y - x} \;=\; f'(x) \quad \text{exists and is finite for all } x \in [a,b].$$

For example, $x^{3/2}$ is differentiable everywhere on $[0,1]$, and $x^{1/2}$ is differentiable everywhere on $[0,1]$ except at $x = 0$.

Exercise B.73. Prove the following.

(a) If f is Lipschitz on $[a,b]$, then f is uniformly continuous and has bounded variation, and $V[f;a,b] \le C\,(b-a)$.

(b) A Lipschitz function need not be differentiable everywhere on $[a,b]$.

(c) If f is differentiable everywhere on $[a,b]$ and f' is bounded on $[a,b]$, then f is Lipschitz with $C = \|f'\|_\infty$. In particular, if f, f' are both continuous on $[a,b]$, then f is Lipschitz.

Not every function of bounded variation need be Lipschitz, see Exercise B.85 below.

Here are some basic properties of the variation function $V[f;a,b]$.

Exercise B.74. Let $f\colon [a,b] \to \mathbb{C}$ be given.

(a) Show that if Γ' is a refinement of Γ, then $S_\Gamma \le S_{\Gamma'}$.

(b) Show that if $[a',b'] \subseteq [a,b]$, then $V[f;a',b'] \le V[f;a,b]$.

(d) Show that if $a < c < b$, then $V[f;a,b] = V[f;a,c] + V[f;c,b]$.

B.8.2 The Jordan Decomposition

Our next goal is to prove the Jordan decomposition, which characterizes real-valued functions of bounded variation as the difference of two monotone increasing functions.

First we introduce the positive and negative variation functions.

Definition B.75. Let $f\colon [a,b] \to \mathbb{R}$ be given. Given a finite partition $\Gamma = \{a = x_0 < \cdots < x_n = b\}$ of $[a,b]$, define

$$S_\Gamma^+ \;=\; \sum_{i=1}^n \bigl[f(x_i) - f(x_{i-1})\bigr]^+ \qquad \text{and} \qquad S_\Gamma^- \;=\; \sum_{i=1}^n \bigl[f(x_i) - f(x_{i-1})\bigr]^-.$$

Thus S_Γ^+ is the sum of the positive terms of S_Γ, and $-S_\Gamma^-$ is the sum of the negative terms. The *positive variation* of f on $[a,b]$ is

$$V^+[f;a,b] \;=\; \sup\bigl\{S_\Gamma^+ : \Gamma \text{ is a partition of } [a,b]\bigr\},$$

and the *negative variation* is

$$V^-[f;a,b] \;=\; \sup\bigl\{S_\Gamma^- : \Gamma \text{ is a partition of } [a,b]\bigr\}.$$

Observe that $S_\Gamma^+ + S_\Gamma^- = S_\Gamma$ and $S_\Gamma^+ - S_\Gamma^- = f(b) - f(a)$. The next exercise extends these equalities from particular partitions to the variation functions.

Exercise B.76. Show that if $f\colon [a,b] \to \mathbb{R}$, then

$$V^+[f;a,b] + V^-[f;a,b] \;=\; V[f;a,b].$$

Further, if any one of $V[f;a,b]$, $V^+[f;a,b]$, or $V^-[f;a,b]$ is finite, then they are all finite, and in this case we also have

$$V^+[f;a,b] - V^-[f;a,b] \;=\; f(b) - f(a).$$

Now we can prove the Jordan decomposition.

Theorem B.77 (Jordan Decomposition). *If $f\colon [a,b] \to \mathbb{R}$ is given, then the following statements are equivalent.*

(a) $f \in \mathrm{BV}[a,b]$.

(b) *There exist monotone increasing functions $f_1,\ f_2\colon [a,b] \to \mathbb{R}$ such that $f = f_1 - f_2$.*

Proof. (a) $\Rightarrow$ (b). For $x \in [a,b]$, the functions $V^+[f;a,x]$ and $V^-[f;a,x]$ are monotonically increasing with x. Furthermore, by Exercise B.76 we have

$$V^+[f;a,x] - V^-[f;a,x] \;=\; f(x) - f(a).$$

Therefore $f = f_1 - f_2$ where $f_1 = V^+[f;a,x] + f(a)$ and $f_2 = V^-[f;a,x]$, and $f_1,\ f_2$ are each monotonically increasing. □

Consequently, a complex-valued function $f\colon [a,b] \to \mathbb{C}$ will have bounded variation if and only if we can write $f = (f_1^+ - f_1^-) + i(f_2^+ - f_2^-)$ where f_1^+, f_1^-, f_2^+, f_2^- are monotone increasing.

B.8.3 Differentiability of Functions of Bounded Variation

An important property of monotone increasing functions is that they are differentiable a.e.

Theorem B.78. *If $f\colon [a,b] \to \mathbb{R}$ is monotone increasing, then $f'(x)$ exists for a.e. x (in fact, for all but at most countably many x), f' is measurable and nonnegative a.e., and*

$$0 \;\le\; \int_a^b f' \;\le\; f(b) - f(a). \tag{B.11}$$

Proof. Note that f is bounded by hypothesis, and extend f to $\mathbb{R}$ by setting $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$.

Since f is increasing and bounded, $f(x-) = \lim_{y \to x^-} f(y)$ and $f(x+) = \lim_{y \to x^+} f(y)$ exist for each x. Hence each point of discontinuity of f must be a jump discontinuity. Further, since f is bounded and increasing, given any fixed $k \in \mathbb{N}$, the set of x such that

$$f(x+) - f(x-) \;\ge\; \frac{1}{k}$$

must be finite. Since every jump discontinuity must satisfy this inequality for some $k \in \mathbb{N}$, we conclude that there can be at most countably many discontinuities.

The proof that f is differentiable almost everywhere is somewhat more involved and will be omitted, see [Fol99] or [WZ77] for details. Assuming this, we have that the functions

$$f_k(x) \;=\; \frac{f(x+1/k) - f(x)}{1/k} \;=\; k\bigl(f(x+\tfrac{1}{k}) - f(x)\bigr)$$

converge pointwise to $f'(x)$ a.e. as $k \to \infty$. In particular, $f' \ge 0$ a.e., and applying Fatou's Lemma we have

$$\begin{aligned}
\int_a^b f' \;&\le\; \liminf_{k\to\infty} \int_a^b f_k \\
&=\; k\int_{a+\frac{1}{k}}^{b+\frac{1}{k}} f \;-\; k\int_a^b f \\
&=\; k\int_{b}^{b+\frac{1}{k}} f \;-\; k\int_a^{a+\frac{1}{k}} f \\
&\le\; k\int_{b}^{b+\frac{1}{k}} f(b) \;-\; k\int_a^{a+\frac{1}{k}} f(a) \qquad \text{(since } f \text{ is increasing)} \\
&=\; f(b) - f(a). \quad \square
\end{aligned}$$

Exercise B.81 shows that equality need not hold in equation (B.11).

Combining Theorem B.78 with the Jordan decomposition, we see that all functions of bounded variation on $[a, b]$ are differentiable a.e. and have integrable derivatives.

Corollary B.79. *If $f \in \mathrm{BV}[a, b]$, then $f'(x)$ exists for a.e. x, and $f' \in L^1[a, b]$.*

The final result stated in this section gives a useful connection between f' and the total variation function.

Theorem B.80. *Given $f \in \mathrm{BV}[a, b]$, set $V(x) = V[f; a, x]$ for $x \in [a, b]$. Then $V'(x) = |f'(x)|$ for almost every $x \in [a, b]$.*

Additional Problems

B.20. Show that if f, $g \in \mathrm{BV}[a, b]$, then $\alpha f + \beta g \in \mathrm{BV}[a, b]$ for all α, $\beta \in \mathbb{C}$ (so $\mathrm{BV}[a, b]$ is a vector space), and $fg \in \mathrm{BV}[a, b]$. If $|g(x)| \geq \varepsilon > 0$ for all $x \in [a, b]$ then $f/g \in \mathrm{BV}[a, b]$.

B.21. Set $f(x) = x^2 \sin(1/x)$ and $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$, and $f(0) = g(0) = 0$. Show that f and g are differentiable everywhere, and $f \in \mathrm{BV}[-1, 1]$ but $g \notin \mathrm{BV}[-1, 1]$ (compare Problem B.23).

B.22. Let $E \subseteq \mathbb{R}$ be measurable, and suppose that $f\colon E \to \mathbb{R}$ is Lipschitz on E, i.e., $|f(x) - f(y)| \leq C\,|x - y|$ for all x, $y \in E$. Prove that if $A \subseteq E$, then $|f(A)|_e \leq C\,|A|_e$ (compare Lemma B.94).

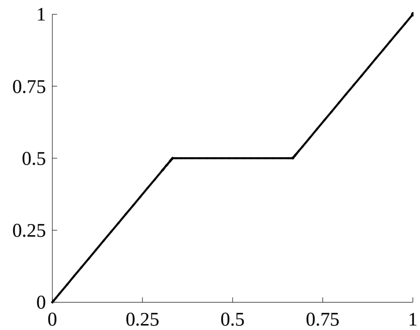
B.9 Absolutely Continuous and Singular Functions

In this section we review the properties of absolutely continuous functions on the real line (which are those functions for which the Fundamental Theorem of Calculus holds) and singular functions on $\mathbb{R}$ (which are differentiable at almost every point but have the property that the derivative is zero a.e.).

B.9.1 Singular Functions on the Real Line

We begin with an example of a singular function.

Exercise B.81 (Cantor–Lebesgue Function). Consider the two functions φ_1, φ_2 pictured in Figure B.1. The function φ_1 takes the constant value $1/2$ on the interval $(1/3, 2/3)$ that is removed in the first stage of the construction of the Cantor middle-thirds set, and is linear on the remaining intervals. The function φ_2 also takes the same constant $1/2$ on the interval $(1/3, 2/3)$ but additionally is constant with values $1/4$ and $3/4$ on the two intervals that are removed in the second stage of the construction of the Cantor set. Continue this process, defining $\varphi_3, \varphi_4, \ldots$, and prove the following facts.



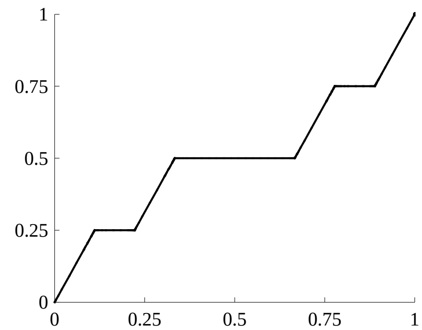


Fig. B.1. First stages in the construction of the Cantor–Lebesgue function. Left: The function φ_1. Right: The function φ_2.

(a) Each φ_k is monotone increasing on $[0, 1]$.

(b) $|\varphi_{k+1}(x) - \varphi_k(x)| < 2^{-k}$ for every $x \in [0, 1]$.

(c) $\varphi(x) = \lim_{k\to\infty} \varphi_k(x)$ converges uniformly on $[0, 1]$.

The function φ constructed in this manner is called the *Cantor–Lebesgue function* or, more picturesquely, the *Devil's staircase*. Prove the following facts.

(d) φ is continuous and monotone increasing on $[0, 1]$, but φ is not Lipschitz.

(e) φ is differentiable for a.e. $x \in [0, 1]$, and $\varphi'(x) = 0$ a.e.

(f) The Fundamental Theorem of Calculus does not apply to φ:

$$\varphi(1) - \varphi(0) \neq \int_0^1 \varphi'(x)\, dx.$$

If we extend φ to $\mathbb{R}$ by reflecting it about the point $x = 1$ and declaring it to be zero outside of $[0, 2]$, we obtain the continuous function φ pictured in Figure B.2. It is interesting that it can be shown that φ is an example of a *refinable function*, as it satisfies the following *refinement equation*:

$$\varphi(x) = \frac{1}{2}\varphi(3x) + \frac{1}{2}\varphi(3x-1) + \varphi(3x-2) + \frac{1}{2}\varphi(3x-3) + \frac{1}{2}\varphi(3x-4). \quad \text{(B.12)}$$

Thus φ equals a finite linear combination of compressed and translated copies of itself, and so exhibits a type of self-similarity. Another example of a refinable function is discussed in Section B.10. Refinable functions are widely studied and play important roles in wavelet theory and in subdivision schemes in computer-aided graphics, see [Dau92].

Exercise B.82. Refinable functions are easy to plot to any desired level of accuracy. For example, since we know the values of $\varphi(k)$ for k integer, we

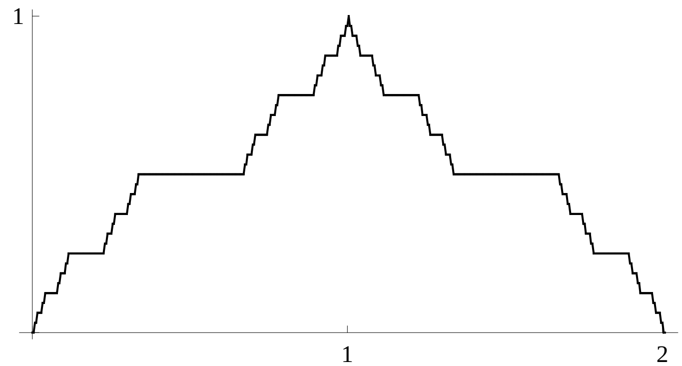


Fig. B.2. The reflected Devil's staircase (Cantor–Lebesgue function).

can compute the values $\varphi(k/3)$ for $k \in \mathbb{Z}$ by considering $x = k/3$ in equation (B.12). Iterating this, we can obtain the values $\varphi(k/3^j)$ for any $k \in \mathbb{Z}$, $j \in \mathbb{N}$. Plot the Cantor–Lebesgue function.

The Cantor–Lebesgue function is the prototypical example of a singular function.

Definition B.83 (Singular Function). A function $f\colon [a,b] \to [-\infty,\infty]$ or $f\colon \mathbb{R} \to \mathbb{C}$ is *singular* if f is differentiable at almost every point in its domain and $f' = 0$ a.e.

B.9.2 Absolutely Continuous Functions on the Real Line

Now we turn to absolutely continuous functions. A collection of intervals in $\mathbb{R}$ is said to be *nonoverlapping* if the interiors of the intervals are disjoint.

Definition B.84 (Absolutely Continuous Function). We say that a function $f\colon [a,b] \to \mathbb{C}$ is *absolutely continuous on* $[a,b]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite or countably infinite collection of nonoverlapping subintervals $\big\{[a_j,b_j]\big\}_j$ of $[a,b]$, we have

$$\sum_j (b_j - a_j) < \delta \quad\implies\quad \sum_j |f(b_j) - f(a_j)| < \varepsilon.$$

We denote the class of absolutely continuous functions on $[a,b]$ by

$$\mathrm{AC}[a,b] \;=\; \big\{f\colon [a,b] \to \mathbb{C} : f \text{ is absolutely continuous on } [a,b]\big\}.$$

The space of *locally absolutely continuous functions on* $\mathbb{R}$ is

$$\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}) \;=\; \big\{f\colon \mathbb{R} \to \mathbb{C} : f \in \mathrm{AC}[a,b] \text{ for every } a < b\big\}.$$

Exercise B.85. Prove the following statements.

(a) If $g \in \mathrm{AC}[a,b]$, then g is uniformly continuous on $[a,b]$.

(b) $\mathrm{Lip}[a,b] \subsetneq \mathrm{AC}[a,b] \subsetneq \mathrm{BV}[a,b]$.

Absolute continuity and singularity are complementary properties, in the sense that only constant functions can be both absolutely continuous and singular. The standard proof of this fact relies on the Vitali Covering Theorem and will be omitted (see Problem B.26 for a different proof).

Theorem B.86. *If $f\colon [a,b] \to \mathbb{C}$ is both absolutely continuous and singular, then f is constant.*

In order to give the connection between absolutely continuous functions and the Fundamental Theorem of Calculus, we need the following important result, whose standard proof relies on the Hardy–Littlewood maximal function and will be omitted. While stated for functions on $\mathbb{R}$, it also applies to functions on finite intervals $[a,b]$, since any integrable function on $[a,b]$ can be extended to a locally integrable function on $\mathbb{R}$, e.g., by declaring it to be zero outside of $[a,b]$.

Theorem B.87 (Lebesgue Differentiation Theorem). *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be given, i.e., f is integrable on every compact subset of $\mathbb{R}$. Then for almost every $x \in \mathbb{R}$,*

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(y)\,dy \;=\; f(x).$$

Consequently, the indefinite integral of f,

$$F(x) \;=\; \int_a^x f(y)\,dy,$$

is differentiable a.e., and $F' = f$ a.e.

In fact, a stronger conclusion holds: If $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, then for a.e. x we have

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} |f(y)-f(x)|\,dy \;=\; \lim_{h\to 0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)-f(x)|\,dy \;=\; 0. \quad \text{(B.13)}$$

Moreover, the intervals $[x, x+h]$ or $[x-h, x+h]$ can be replaced by any collection of sets $\{S_h\}_{h>0}$ that *shrink regularly* to x, which means that $\mathrm{diam}(S_h) \to 0$, and there exists a constant $C > 0$ such that if Q_h is the smallest interval centered at x that contains S_h, then $|Q_h| \le C\,|S_h|$.

Definition B.88 (Lebesgue Set). Given $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, a point x for which equation (B.13) holds is called a *Lebesgue point* of f, and the set of Lebesgue points is called the *Lebesgue set* of f.

Thus, almost every point in the domain of a locally integrable function is a Lebesgue point.

Exercise B.89. Given $f \in L^1_{\text{loc}}(\mathbb{R})$, show that every point of continuity of f is a Lebesgue point of f.

The next exercise is motivation for the *Fundamental Theorem of Calculus* for absolutely continuous functions. This exercise shows that the antiderivative of an integrable function is absolutely continuous.

Exercise B.90. Show that if $f \in L^1[a,b]$, then $g(x) = \int_a^x f(t)\,dt$ belongs to $\mathrm{AC}[a,b]$, and furthermore $g' = f$ a.e.

In fact, much more holds.

Theorem B.91 (Fundamental Theorem of Calculus). *If $g\colon [a,b] \to \mathbb{C}$, then the following statements are equivalent.*

(a) $g \in \mathrm{AC}[a,b]$.

(b) *There exists $f \in L^1[a,b]$ such that*

$$g(x) - g(a) \;=\; \int_a^x f(t)\,dt, \qquad x \in [a,b].$$

(c) *g is differentiable almost everywhere, $g' \in L^1[a,b]$, and*

$$g(x) - g(a) \;=\; \int_a^x g'(t)\,dt, \qquad x \in [a,b].$$

Proof. (a) $\Rightarrow$ (c). Suppose that g is absolutely continuous on $[a,b]$. Then g has bounded variation, and so by Corollary B.79 we know that g' exists a.e. and is integrable. Therefore the function

$$G(x) \;=\; \int_a^x g'$$

is well-defined for each $x \in [a,b]$. Moreover, by the Lebesgue Differentiation Theorem, $G' = g'$ a.e. Hence $(G-g)' = 0$ a.e., so the function $G-g$ is singular on $[a,b]$. On the other hand, both g and G are absolutely continuous on $[a,b]$, so $G-g$ is absolutely continuous as well. Therefore we have by Theorem B.86 that $G-g$ is constant. Consequently, given any $x \in [a,b]$, we have

$$G(x) - g(x) \;=\; G(a) - g(a) \;=\; 0 - g(a) \;=\; -g(a).$$

Thus $G(x) = g(x) - g(a)$ for all $x \in [a,b]$, so statement (c) holds. □

If φ is the Cantor–Lebesgue function on $[0,1]$, then φ is singular, and hence is differentiable almost everywhere with $\varphi' \in L^1[a,b]$, yet we have $\varphi(x) - \varphi(0) \neq \int_0^x \varphi'(t)\,dt = 0$, confirming the fact that φ is not absolutely continuous.

We can use Exercise B.90 and Theorem B.86 to prove the following fundamental decomposition of functions of bounded variation.

Corollary B.92. *If $f \in \mathrm{BV}[a,b]$, then $f = g + h$ where $g \in \mathrm{AC}[a,b]$ and h is singular on $[a,b]$. Moreover, g and h are unique up to additive constants, and we can take*

$$g(x) = \int_a^x f', \qquad x \in [a,b]. \tag{B.14}$$

Proof. Since f has bounded variation on $[a,b]$, we know that f' exists a.e. and is integrable. Therefore the function g given by equation (B.14) is well-defined. Set $h = f - g$. By Exercise B.90, we have $g \in \mathrm{AC}[a,b]$ and $g' = f'$ a.e., so $h' = 0$ a.e. Thus h is singular.

If we also had $f = g_1 + h_1$ with g_1 absolutely continuous and h_1 singular, then $g - g_1 = h_1 - h$, so $g - g_1$ and $h_1 - h$ are each absolutely continuous and singular, and therefore are constant by Theorem B.86. □

An important fact is that integration by parts is valid for absolutely continuous functions.

Exercise B.93 (Integration by Parts). Show that if f, $g \in \mathrm{AC}[a,b]$, then

$$\int_a^b f(x)\, g'(x)\, dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)\, g(x)\, dx.$$

B.9.3 Preparation for the Banach–Zarecki Theorem

The Banach–Zarecki Theorem provides some reformulations of absolute continuity. To prove it, we will need two lemmas, and in order to motivate these we recall Exercise B.73, which states that if $f\colon [a,b] \to \mathbb{C}$ is differentiable everywhere and f' is bounded, then f is Lipschitz and hence is absolutely continuous on $[a,b]$. One of the implications of the Banach–Zarecki Theorem is the much more subtle fact that if f is differentiable everywhere on $[a,b]$ and we assume only that $f' \in L^1[a,b]$, then f is absolutely continuous. We will make use of this to prove Theorem 1.47 in Chapter 1. The subtlety here is that while the assumptions f, $f' \in L^1[a,b]$ do imply that the antiderivative $g(x) = \int_a^x f'(t)\, dt$ exists and is absolutely continuous, it is not at all obvious whether g need equal f.

Our first lemma is a refinement of Problem B.22, which showed that if a function f is Lipschitz on $[a,b]$ and E is any subset of $[a,b]$, then $|f(E)|_e \le C\,|E|_e$, where C is a Lipschitz constant for f. In particular, if f is differentiable on $[a,b]$ and f' is bounded on $[a,b]$, then we know that f is Lipschitz on every subset of $[a,b]$, and hence can apply Problem B.22 to this f. However, if we only know that f is differentiable and f' is bounded *on a particular subset E* then we cannot apply Problem B.22. Still, by making the argument a little more sophisticated, we can show that the same conclusion holds.

Lemma B.94. *Let $f\colon [a,b] \to \mathbb{R}$ and $E \subseteq [a,b]$ be given. Suppose that f is differentiable at every point of E, and that*

$$M = \sup_{x \in E} |f'(x)| < \infty.$$

Then

$$|f(E)|_e \le M\,|E|_e.$$

Proof. Fix $\varepsilon > 0$. Given $x \in E$, we have

$$\lim_{\substack{y \to x, \\ y \in [a,b]}} \frac{|f(x) - f(y)|}{|x - y|} = |f'(x)| \le M.$$

Hence there exists some $n_x \in \mathbb{N}$ such that if $y \in [a,b]$ then

$$|x - y| < \frac{1}{n_x} \implies |f(x) - f(y)| \le (M + \varepsilon)\,|x - y|.$$

Therefore, if for each $n \in \mathbb{N}$ we define

$$E_n = \Bigl\{x \in E : \text{if } y \in [a,b] \text{ and } |x-y| < \frac{1}{n} \text{ then } |f(x)-f(y)| \le (M+\varepsilon)\,|x-y|\Bigr\},$$

then we have that $E = \cup E_n$. Further, $E_1 \subseteq E_2 \subseteq \cdots$. Even though the sets E_n need not measurable, we have by Problem B.3 that continuity from below holds for *exterior* Lebesgue measure, so

$$|E|_e = \lim_{n \to \infty} |E_n|_e.$$

As the sets $f(E_n)$ are also nested increasing and increase to $f(E)$, we also have

$$|f(E)|_e = \lim_{n \to \infty} |f(E_n)|_e.$$

Now, for each n, we can find at most countably many intervals I_n^k such that

$$E_n \subseteq \bigcup_k I_n^k \qquad \text{and} \qquad \sum_k |I_n^k| \le |E_n|_e + \varepsilon.$$

By subdividing if necessary, we may assume that each interval I_n^k has length less than $\frac{1}{n}$. Therefore, if we take $x, y \in E_n \cap I_n^k$, then we have $|x - y| < \frac{1}{n}$, so

$$|f(x) - f(y)| \le (M + \varepsilon)\,|x - y|.$$

Consequently, the image $f(E_n \cap I_n^k)$ is contained in an interval of length at most $(M + \varepsilon)\,|I_n^k|$, so

$$|f(E_n \cap I_n^k)|_e \le (M + \varepsilon)\,|I_n^k|.$$

Therefore

$$|f(E_n)|_e \le \sum_k |f(E_n \cap I_n^k)| \le (M + \varepsilon) \sum_k |I_n^k| \le (M + \varepsilon)\,(|E_n|_e + \varepsilon).$$

Hence,

$$|f(E)|_e = \lim_{n\to\infty} |f(E_n)|_e \le (M+\varepsilon) \lim_{n\to\infty} (|E_n|_e + \varepsilon) = (M+\varepsilon)\,(|E|_e + \varepsilon).$$

Since ε is arbitrary, the result follows. □

The hypothesis of differentiability in Lemma B.94 can be weakened, see [BBT97, Lemma 7.9].

The second lemma relates the exterior measure of $f(E)$ to the integral of $|f'|$ on E. Note that even though we now assume that E is measurable, we cannot conclude that $f(E)$ is measurable, and hence this result must also be formulated in terms of the exterior Lebesgue measure of $f(E)$ (compare Problem B.24, which shows that an absolutely continuous function must map measurable sets to measurable sets, but an arbitrary continuous function need not do so).

Lemma B.95. *Let $f\colon [a,b] \to \mathbb{R}$ be measurable. If $E \subseteq [a,b]$ is measurable and f is differentiable at every point of E, then*

$$|f(E)|_e \le \int_E |f'|.$$

Proof. By Problem B.6, the derivative f' is a measurable function, and hence $\int_E |f'|$ does exist as an extended real number. For each $k \in \mathbb{N}$, define

$$E_k = \{x \in E : (k-1)\varepsilon \le |f'(x)| < k\varepsilon\}.$$

Since f is differentiable everywhere on E, we have $E = \cup E_k$ disjointly. Further, by Lemma B.94 we have that $|f(E_k)|_e \le k\varepsilon\,|E_k|$. Therefore

$$\begin{aligned} |f(E)|_e = \left|\bigcup_{k=1}^{\infty} f(E_k)\right|_e &\le \sum_{k=1}^{\infty} |f(E_k)|_e \\ &\le \sum_{k=1}^{\infty} k\varepsilon\,|E_k| \\ &= \sum_{k=1}^{\infty} (k-1)\varepsilon\,|E_k| + \sum_{k=1}^{\infty} \varepsilon\,|E_k| \\ &\le \sum_{k=1}^{\infty} \int_{E_k} |f'| + \varepsilon\,|E| \\ &= \int_E |f'| + \varepsilon\,|E|. \end{aligned}$$

Since ε is arbitrary, the result follows. □

B.9.4 The Banach–Zarecki Theorem and its Implications

Now we prove the Banach–Zarecki Theorem.

Theorem B.96 (Banach–Zarecki Theorem). *Given $f\colon [a,b] \to \mathbb{C}$, the following statements are equivalent.*

(a) $f \in \mathrm{AC}[a,b]$.

(b) *f is continuous, $f \in \mathrm{BV}[a,b]$, and $|f(A)| = 0$ for every $A \subseteq [a,b]$ with $|A| = 0$.*

(c) *f is continuous and is differentiable a.e., $f' \in L^1[a,b]$, and $|f(A)| = 0$ for every $A \subseteq [a,b]$ with $|A| = 0$.*

Proof. By breaking into real and imaginary parts, it suffices to prove the result for real-valued functions.

(a) $\Rightarrow$ (b). Suppose that $f \in \mathrm{AC}[a,b]$ is real-valued. Then f is continuous and has bounded variation by Exercise B.85, so it only remains to show that f maps zero measure sets to zero measure sets.

Suppose that A is a subset of $[a,b]$ with measure zero. Since $\{f(a), f(b)\}$ is a set of measure zero, it suffices to assume that $A \subseteq (a,b)$ with $|A| = 0$. Choose any $\varepsilon > 0$. Since f is absolutely continuous, there exists a $\delta > 0$ such that if $\{[a_j,b_j]\}_j$ is any collection of nonoverlapping intervals such that $\sum (b_j - a_j) < \delta$, then $\sum |f(b_j) - f(a_j)| < \varepsilon$.

By Theorem B.5, we can find an open set $U \supseteq A$ with measure $|U| < |A| + \varepsilon = \varepsilon$, and by intersecting with the open interval (a,b) we may assume that $U \subseteq (a,b)$. Then we can write $U = \cup (a_j,b_j)$ as a union of at most countably many disjoint open intervals contained in (a,b). Since $[a_j,b_j] \subseteq [a,b]$, there is a point $c_j \in [a_j,b_j]$ where f attains its minimum value on $[a_j,b_j]$, and likewise a point $d_j \in [a_j,b_j]$ where f attains its maximum. Then we have $\sum |d_j - c_j| \le \sum (b_j - a_j) < \delta$, so

$$|f(A)|_e \;\le\; |f(U)|_e \;\le\; \sum_j \bigl|f[a_j,b_j]\bigr|_e \;\le\; \sum_j \bigl|f(d_j) - f(c_j)\bigr| \;<\; \varepsilon.$$

Since ε is arbitrary, we conclude that $|f(A)| = 0$.

(c) $\Rightarrow$ (a). Assume that statement (c) holds for some real-valued f. Let D be the set of points where f is differentiable, so $Z = [a,b] \backslash D$ has measure zero.

Suppose that $[c,d]$ is any subinterval of $[a,b]$. By the Intermediate Value Theorem, $f[c,d]$ contains either $[f(c), f(d)]$ or $[f(d), f(c)]$, depending on order. Let $E = [c,d] \cap D$ and $F = [c,d] \backslash D$. Then F has measure zero, so by hypothesis we have $|f(F)| = 0$. Also, f is differentiable everywhere on E, so by Lemma B.95 we have

$$\begin{aligned} |f(d) - f(c)| \;\le\; |f([c,d])|_e \;\le\; |f(E)|_e \;+\; |f(F)|_e \\ \le\; \int_E |f'| \;+\; 0 \;=\; \int_c^d |f'|, \end{aligned} \tag{B.15}$$

the final equality following from the fact that E is a subset of $[c,d]$ with full measure.

Now choose any $\varepsilon > 0$. Since f' is integrable, by Problem B.8 there exists a $\delta > 0$ such that $\int_E |f'| < \varepsilon$ for any measurable set $E \subseteq [a,b]$ with $|E| < \delta$. Suppose that $\{[a_j,b_j]\}_j$ is a collection of finitely or countably many nonoverlapping intervals in $[a,b]$ such that $\sum (b_j - a_j) < \delta$. Define $E = \cup [a_j,b_j]$, so $|E| < \delta$. Then by equation (B.15), we have

$$\sum_j \bigl|f(b_j) - f(a_j)\bigr| \;\le\; \sum_j \int_{a_j}^{b_j} |f'| \;=\; \int_E |f'| \;<\; \varepsilon,$$

so f is absolutely continuous on $[a,b]$. □

As a corollary, we obtain the following fact that will be of use to us in Chapter 1 (see Theorem 1.47).

Corollary B.97. *If* $f\colon [a,b] \to \mathbb{C}$ *is everywhere differentiable and* $f' \in L^1[a,b]$, *then* $f \in \mathrm{AC}[a,b]$.

Proof. By splitting into real and imaginary parts, we may assume that f is real-valued. Suppose that $A \subseteq [a,b]$ and $|A| = 0$. Then since f is differentiable at every point of A, we have by Lemma B.95 that $|f(A)|_e \le \int_A |f'| = 0$. Theorem B.96 therefore implies that f is absolutely continuous. □

The hypotheses of Corollary B.97 can be relaxed somewhat. For example, if f is differentiable except at *countably* many points and $f' \in L^1[a,b]$, then f will be absolutely continuous (Problem B.28). However, the assumptions that f is differentiable a.e. and $f' \in L^1[a,b]$ are by themselves not sufficient to ensure that f is absolutely continuous (consider the Cantor–Lebesgue function).

Additional Problems

B.23. Define $g(x) = x^2 \sin(1/x^2)$ for $x \ne 0$ and $g(0) = 0$. Show that $g \in L^1[-1,1]$, g is everywhere differentiable, $g' \notin L^1[-1,1]$, and $g \notin \mathrm{AC}[-1,1]$ (compare Problem B.21).

B.24. (a) Show that a continuous function need not map a measurable set to a measurable set.

(b) Show that if $f\colon [a,b] \to \mathbb{C}$ is absolutely continuous and $E \subseteq [a,b]$ is measurable, then $|f(E)|$ is measurable as well.

B.25. Show that $\mathrm{AC}[a,b]$ is closed under pointwise products, i.e., if $f,\, g \in \mathrm{AC}[a,b]$ then $fg \in \mathrm{AC}[a,b]$.

B.26. Show that if $f \in \mathrm{AC}[a,b]$ and $f' = 0$ a.e. then f is constant.

B.27. Suppose that $f \in \mathrm{AC}_{\mathrm{loc}}[a,b]$. In particular, f' exists a.e. Suppose there is a continuous function g such that $f' = g$ a.e. Show that f is differentiable everywhere and $f'(x) = g(x)$ for all $x \in [a,b]$.

B.28. Suppose that $f\colon [a,b] \to \mathbb{C}$ is differentiable at all but countably many points, and $f' \in L^1[a,b]$. Show that f is absolutely continuous.

B.10 Hölder Continuity

Hölder continuity is a generalization of Lipschitz continuity (Definition B.71).

Definition B.98. We say that a function $f\colon \mathbb{R} \to \mathbb{C}$ is *Hölder continuous* with exponent $\alpha > 0$ if there exists a constant $K > 0$ such that

$$\forall\, x, y \in \mathbb{R}, \quad |f(x) - f(y)| \;\le\; K\, |x-y|^{\alpha}.$$

Thus, Lipschitz continuity is Hölder continuity for the case $\alpha = 1$. The reader should verify that the only functions that are Hölder continuous with exponent $\alpha > 1$ are the constant functions.

While a Lipschitz function is "almost differentiable" in some sense, the graph of a function that is Hölder continuous with exponent $\alpha < 1$ typically has a "fractal" appearance. The smaller that we must take α, the more jagged the graph of the function appears.

One example of a Hölder continuous function is the Cantor–Lebesgue function.

Exercise B.99. Show that the Cantor–Lebesgue function is Hölder continuous precisely for exponents α in the range $0 < \alpha \le \log_3 2 \approx 0.6309\ldots$.

Another example is the *Daubechies D_4 function*. This is the compactly supported function that satisfies the four-term *refinement equation*

$$\begin{aligned} D_4(x) \;=\; & \frac{1+\sqrt{3}}{4} D_4(2x) + \frac{3+\sqrt{3}}{4} D_4(2x-1) \\ & + \frac{3-\sqrt{3}}{4} D_4(2x-2) + \frac{1-\sqrt{3}}{4} D_4(2x-3). \end{aligned} \tag{B.16}$$

Thus, D_4 exhibits a kind of self-similarity, as it equals a linear combination of four smaller, shifted copies of itself. It can be shown that there exists a unique (up to scale) compactly supported function that satisfies this refinement equation, and furthermore this solution is Hölder continuous precisely for those exponents α that lie in the range

$$0 \;<\; \alpha \;\le\; -\log_2\Bigl(\frac{1+\sqrt{3}}{4}\Bigr) \;\approx\; 0.5500\ldots,$$

see [Dau92]. The Cantor–Lebesgue function also satisfies a refinement equation but based on dilation by 3 rather than 2, see equation (B.12).

Exercise B.100. For this exercise, assume that equation (B.16) has a solution that is continuous and compactly supported.

(a) Show that $\mathrm{supp}(D_4) \subseteq [0,3]$.

(b) Combine part (a) with the refinement equation to find the values of $D_4(k)$ for all integer k.

(c) Now compute the values $D_4(k/2)$ for $k \in \mathbb{Z}$ by considering $x = k/2$ in equation (B.16). Iterating this, we can obtain the values $D_4(k/2^j)$ for any $k \in \mathbb{Z}$, $j \in \mathbb{N}$. Plot the Daubechies D_4 function.

When suitably normalized, D_4 has the interesting property that its integer translates are orthonormal, i.e., $\{D_4(x-k)\}_{k\in\mathbb{Z}}$ forms an orthonormal system in $L^2(\mathbb{R})$. The Daubechies D_4 function is but one refinable function that has orthonormal integer translates. Each such *orthonormal scaling function* leads to a second function, called a *wavelet*, which generates an orthonormal basis for *all* of $L^2(\mathbb{R})$ via the operations of translation and dilation. For D_4, the corresponding wavelet is

$$\begin{aligned} W_4(x) \;=\; & \frac{1-\sqrt{3}}{4} D_4(2x) - \frac{3-\sqrt{3}}{4} D_4(2x-1) \\ & + \frac{3+\sqrt{3}}{4} D_4(2x-2) - \frac{1+\sqrt{3}}{4} D_4(2x-3). \end{aligned}$$

This function has the property that $\bigl\{2^{n/2} W_4(2^n x - k)\bigr\}_{n,k\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$, and

$$\bigl\{D_4(x-k)\bigr\}_{k\in\mathbb{Z}} \;\cup\; \bigl\{2^{n/2} W_4(2^n x - k)\bigr\}_{n\geq 0,\, k\in\mathbb{Z}}$$

is another orthonormal basis for $L^2(\mathbb{R})$.

Why the subscript 4? The Daubechies scaling function D_4 is the second of an infinite family of functions $\{D_{2N}\}_{N\in\mathbb{N}}$, each of which satisfies a $2N$-term refinement equation, is supported in $[0, 2N-1]$, and has orthonormal integer translates. Moreover, the smoothness of D_{2N} increases with N. For example, $D_2 = \chi_{[0,1]}$ is discontinuous, D_4 is continuous but not differentiable, and D_6 is just "barely" differentiable. Each of these scaling functions has an associated wavelet W_{2N} whose integer translates and dyadic dilations form an orthonormal basis for $L^2(\mathbb{R})$.

The first wavelet, the function $W_2 = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, was introduced by Alfréd Haar (1885–1933) in his 1910 Ph.D. thesis [Haa10], and is called the *Haar wavelet*, see Problem B.29. It was not until much later that other wavelets, such as the Daubechies scaling functions and wavelets, were discovered. We will not be able to do justice to wavelet theory here, but only mention that the main papers in the development of wavelet theory, including a translation of Haar's original paper into English, appear in the reprint volume [HW06]. More fundamentally, we refer to Daubechies' classic text [Dau92] for complete details on scaling functions and wavelets, and to the texts by Mallat

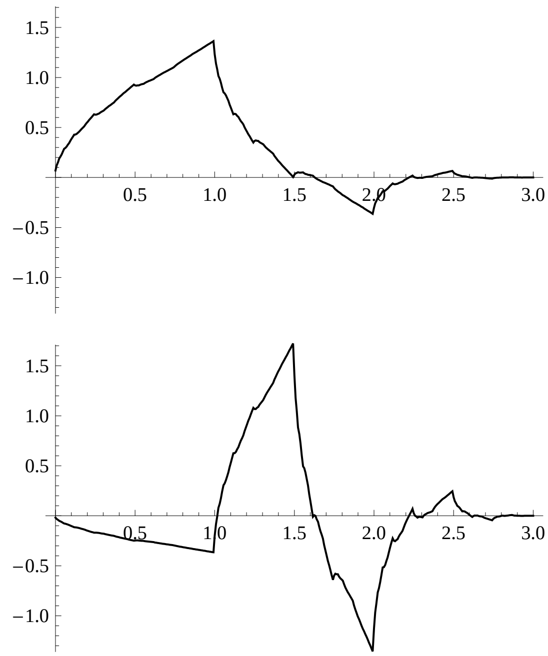


Fig. B.3. Top: The Daubechies D_4 scaling function. Bottom: The corresponding wavelet W_4.

[Mal98] and Strang and Nguyen [SN96] for their relation to signal processing. The text by Walnut [Wal02] is an accessible introduction to wavelet theory and its applications.

Additional Problems

B.29. Let $\chi = \chi_{[0,1)}$ be the box function, and let $\psi = W_2 = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ be the *Haar wavelet*. Prove that

$$\big\{\chi(x-k)\big\}_{k\in\mathbb{Z}} \;\cup\; \big\{2^{n/2}\psi(2^n x - k)\big\}_{n\geq 0,\, k\in\mathbb{Z}}$$

forms an orthonormal basis for $L^2(\mathbb{R})$, called the *Haar system*. Observe that χ satisfies the refinement equation $\chi(x) = \chi(2x) + \chi(2x-1)$, while ψ is determined from χ by the equation $\psi(x) = \chi(2x) - \chi(2x-1)$.

B.30. Given $0 < \alpha < 1$, define

$$C^\alpha(\mathbb{R}) \;=\; \big\{f \in C(\mathbb{R}) : f \text{ is Hölder continuous with exponent } \alpha\big\}.$$

Show that

$$\|f\|_{C^\alpha} \;=\; |f(0)| \;+\; \sup_{x\neq y} \frac{f(x)-f(y)}{|x-y|^\alpha}$$

is a norm on $C^\alpha(\mathbb{R})$, and that $C^\alpha(\mathbb{R})$ is complete with respect to this norm.