E

Topological Vector Spaces

Many of the important vector spaces in analysis have topologies that are generated by a *family of seminorms* instead of a given metric or a norm. We will consider these types of topologies in this appendix. References for the material in this appendix include the texts by Conway [Con90], Folland [Fol99], or Rudin [Rud91].

E.1 Motivation and Examples

If X is a metric space then every open subset of X is, by definition, a union of open balls. The set of open balls is an example of a *base* for the topology on X (see Definition E.9). If X is also a vector space, it is usually very important to know whether these open balls are convex. This is certainly true if the metric is induced from a norm, but it is not true in general.

Example E.1. Exercise B.57 tells us that if $E \subseteq \mathbb{R}$ and $0 < p < 1$, then $L^p(E)$ is a complete metric space with respect to the metric $\mathrm{d}(f,g) = \|f-g\|_p^p$. Because the operations of vector addition and scalar multiplication are continuous on $L^p(E)$, we call $L^p(E)$ a *topological vector space* (see Definition E.12). Unfortunately, the unit ball in $L^p(E)$ is not convex when $p < 1$ (compare the illustration in Figure E.1). In fact, it can be shown that $L^p(E)$ contains *no* nontrivial open convex subsets, so we cannot get around this issue by substituting some other open sets for the open balls. When $p < 1$, there is no base for the topology on $L^p(E)$ that consists of convex sets.

For $0 < p < 1$, the topology on $L^p(E)$ is generated by a metric, but this metric is not induced from a norm or a family of seminorms. We will see that topologies that are generated from families of seminorms do have a base that consists of convex open sets. If the family of seminorms is finite, then we can find a single norm that induces the same topology. If the family of seminorms is countable, then we can find a single metric that induces the same topology.

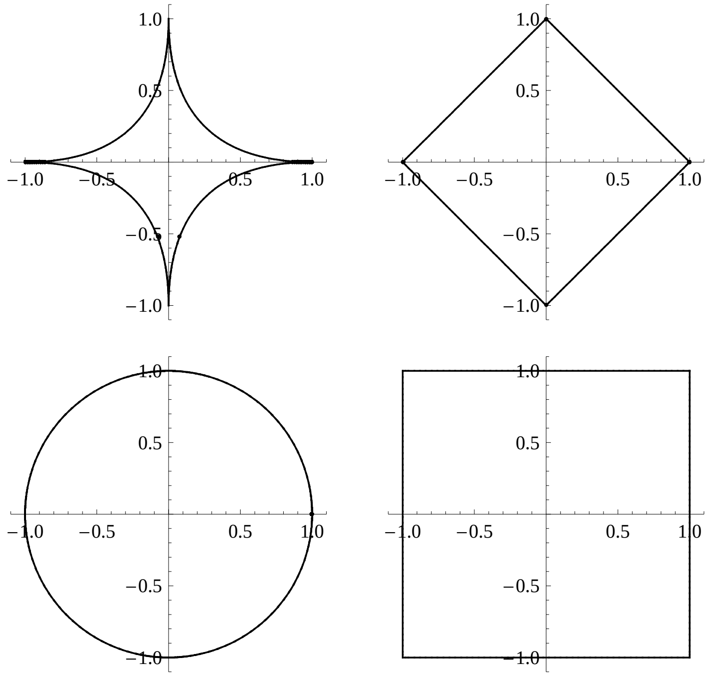

Fig. E.1. Unit circles in $\mathbb{R}^2$ with respect to various metrics. Top left: $\ell^{1/2}$. Top right: ℓ^1. Bottom left: ℓ^2. Bottom right: ℓ^∞.

A significant advantage of having a topology induced from a metric is that the corresponding convergence criterion can be defined in terms of convergence of ordinary sequences instead of convergence of nets (see Section A.7).

In this section we will give several examples of spaces whose topologies are induced from families of seminorms, indicating without proof some of the special features that we will consider in more detail in the following sections. In these examples, we specify a convergence criterion instead of directly specifying a topology. The connection between convergence criteria and topologies is reviewed in Section A.7 and will be expanded on below.

The simplest examples are those where the family of seminorms is finite.

Example E.2. Given $m \geq 0$, Exercise A.21 shows that the space $C_b^m(\mathbb{R})$ is a Banach space with respect to the norm

$$\|f\|_{C_b^m} \;=\; \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^{(m)}\|_\infty.$$

The natural associated family of seminorms is $\{\rho_n\}_{n=0}^m$, where

$$\rho_n(f) \;=\; \|f^{(n)}\|_\infty, \qquad f \in C_b^m(\mathbb{R}).$$

Note that ρ_0 is a norm on $C_b^m(\mathbb{R})$, while $\rho_1, \dots, \rho_m$ are only seminorms. However, while $C_b^m(\mathbb{R})$ is complete with respect to the norm $\|\cdot\|_{C_b^m}$, if $m > 0$ then it is not complete with respect to the norm ρ_0. We need all of the seminorms to define the correct topology. Since there are only finitely many, we usually combine these seminorms to form the norm $\|\cdot\|_{C_b^m}$.

The convergence criterion defined by the norm $\|\cdot\|_{C_b^m}$ is:

$$f_k \to f \text{ in } C_b^m(\mathbb{R}) \qquad \Longleftrightarrow \qquad \|f - f_k\|_{C_b^m} \to 0.$$

The same convergence criterion defined in terms of the family of seminorms working in concert is:

$$f_k \to f \text{ in } C_b^m(\mathbb{R}) \qquad \Longleftrightarrow \qquad \rho_n(f - f_k) \to 0 \;\text{ for each } n = 0, \dots, m.$$

Things become more interesting when there are infinitely many seminorms in the family. In this case, we usually cannot create a single norm that defines the same topology.

Example E.3. The topology on the space $C_b^\infty(\mathbb{R})$ is defined by the countable family of seminorms $\{\rho_n(f)\}_{n=0}^\infty$, where

$$\rho_n(f) \;=\; \|f^{(n)}\|_\infty, \qquad f \in C_b^\infty(\mathbb{R}).$$

The corresponding convergence criterion is joint convergence with respect to all of the seminorms:

$$f_k \to f \text{ in } C_b^\infty(\mathbb{R}) \qquad \Longleftrightarrow \qquad \rho_n(f - f_k) \to 0 \;\text{ for each } n = 0, 1, \dots.$$

Because there are only countably many seminorms, we will see that this topology is induced from the metric

$$\mathrm{d}(f,g) \;=\; \sum_{n=0}^{\infty} 2^{-n} \, \frac{\|f^{(n)} - g^{(n)}\|_\infty}{1 + \|f^{(n)} - g^{(n)}\|_\infty}.$$

While the topology on $L^p(E)$, $p < 1$, is also induced from a metric, that topology has no base consisting of convex open sets, whereas the topology on $C_b^\infty(\mathbb{R})$ does have a natural base consisting of convex open sets.

Another example of a space whose topology is generated by a countable family of seminorms is the Schwartz space. This space plays an important role in harmonic analysis, and appears throughout the main part of this volume.

Example E.4. The *Schwartz space* is

$$\mathcal{S}(\mathbb{R}) \;=\; \bigl\{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R}) \;\text{ for all } m, n \ge 0\bigr\}.$$

The natural associated family of seminorms is $\{\rho_{mn}(f)\}_{m,n\ge 0}$, where

$$\rho_{mn}(f) \;=\; \|x^m f^{(n)}(x)\|_\infty, \qquad f \in \mathcal{S}(\mathbb{R}).$$

Convergence in the Schwartz space is defined by

$$f_k \to f \text{ in } \mathcal{S}(\mathbb{R}) \qquad \Longleftrightarrow \qquad \rho_{mn}(f - f_k) \to 0 \;\text{ for all } m, n \ge 0.$$

As there are only countably many seminorms, there does exist a metric that induces the same topology.

Sometimes, an uncountable family of seminorms can be reduced to a countable family.

Example E.5. The space $L^1_{\mathrm{loc}}(\mathbb{R})$ consisting of all locally integrable functions on $\mathbb{R}$ was introduced in Definition B.62. A natural associated family of seminorms is $\{\rho_K : \text{compact } K \subseteq \mathbb{R}\}$, where

$$\rho_K(f) \;=\; \|f \cdot \chi_K\|_1 \;=\; \int_K |f(x)|\,dx, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}).$$

Since each compact set $K \subseteq \mathbb{R}$ is contained in some interval $[-N, N]$, the same topology is determined by the countable family of seminorms $\{\rho_N\}_{N\in\mathbb{N}}$, where

$$\rho_N(f) \;=\; \|f \cdot \chi_{[-N,N]}\|_1 \;=\; \int_{-N}^{N} |f(x)|\,dx, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}).$$

Since we can reduce to countably many seminorms, this topology is also induced from an associated metric.

The next example combines several of the features of the preceding ones.

Example E.6. Functions in $C^\infty(\mathbb{R})$ need not be bounded, so we cannot create a topology on $C^\infty(\mathbb{R})$ by using the seminorms from Example E.3. Instead, the topology is induced from the family $\{\rho_{K,n} : n \geq 0, \text{ compact } K \subseteq \mathbb{R}\}$, where

$$\rho_{K,n}(f) \;=\; \|f^{(n)} \cdot \chi_K\|_\infty, \qquad f \in C^\infty(\mathbb{R}).$$

As in Example E.5, we can generate the same topology using a countable family of seminorms. T,his space with this topology is often denoted by

$$\mathcal{E}(\mathbb{R}) \;=\; C^\infty(\mathbb{R}).$$

Not every family of seminorms can be reduced to a countable family.

Example E.7 (The Weak Topology). Let X be any normed linear space. The norm induces one topology on X, but it is not the only natural topology. Each element μ of the dual space X^* provides us with a seminorm ρ_μ on X defined by

$$\rho_\mu(x) \;=\; |\langle x, \mu\rangle|, \qquad x \in X.$$

The topology induced by the family of seminorms $\{\rho_\mu\}_{\mu\in X^*}$ is called the *weak topology* on X. Since there are uncountably many seminorms (and no obvious way to reduce to a countable collection in general), in order to relate the topology to a convergence criterion we must use nets instead of sequences. Writing $x_i \xrightarrow{\mathrm{w}} x$ to denote convergence of a net $\{x_i\}_{i\in I}$ in X with respect to the weak topology, the convergence criterion is

$$x_i \xrightarrow{\mathrm{w}} x \quad \iff \quad \rho_\mu(x - x_i) \to 0 \;\text{ for all } \mu \in X^*.$$

Equivalently,

$$x_i \xrightarrow{\text{w}} x \qquad \Longleftrightarrow \qquad \langle x_i, \mu\rangle \to \langle x, \mu\rangle \text{ for all } \mu \in X^*.$$

Norm convergence implies weak convergence, but the converse fails in general. In terms of topologies, every set that is open with respect to the weak topology is also open with respect to the norm topology, but not conversely. In essence, it is "easier" to converge in the weak topology because there are fewer open sets in that topology (hence the weak topology is *weaker* than the norm topology).

Example E.8 (The Weak Topology).* Let X be any normed linear space. Then its dual space X^* is also a normed linear space, and hence has a topology defined by its norm, as well as a weak topology as described above. However, there is also a third natural topology associated with X^*. Each element x in X determines a seminorm ρ_x on X^* by

$$\rho_x(\mu) \;=\; |\langle x, \mu\rangle|, \qquad \mu \in X^*.$$

The topology this family of seminorms $\{\rho_x\}_{x\in X}$ induces is called the *weak* topology* on X^*, and convergence with respect to this topology is denoted by $\mu_i \xrightarrow{\text{w}^*} \mu$. Explicitly, if $\{\mu_i\}_{i\in I}$ is a net in X^* then the convergence criterion corresponding to the weak* topology is

$$\mu_i \xrightarrow{\text{w}^*} \mu \qquad \Longleftrightarrow \qquad \rho_x(\mu - \mu_i) \to 0 \text{ for all } x \in X.$$

Equivalently,

$$\mu_i \xrightarrow{\text{w}^*} \mu \qquad \Longleftrightarrow \qquad \langle x, \mu_i\rangle \to \langle x, \mu\rangle \text{ for all } x \in X.$$

Since $X \subseteq X^{**}$, the family of seminorms associated with the weak* topology includes only some of the seminorms associated with the weak topology. Hence weak convergence in X^* implies weak* convergence in X^*. Of course, if X is reflexive then $X = X^{**}$ and the weak and weak* topologies on X^* are the same.

Additional Problems

E.1. Let $\mathcal{F}(\mathbb{R})$ be the vector space containing all functions $f\colon \mathbb{R} \to \mathbb{C}$. For each $x \in \mathbb{R}$, define a seminorm on $\mathcal{F}(\mathbb{R})$ by $\rho_x(f) = |f(x)|$.

(a) Show that convergence with respect to the family of seminorms $\{\rho_x\}_{x\in\mathbb{R}}$ corresponds to pointwise convergence of functions.

(b) Show that there is no norm on $\mathcal{F}(\mathbb{R})$ that defines the same convergence criterion.

E.2 Topological Vector Spaces

Now we will consider vector spaces that have "nice" topologies, especially those that are generated by families of seminorms.

E.2.1 Base for a Topology

A base is a set of "building blocks" for a topology, playing a role analogous to the one played by the collection of open balls in a metric space.

Definition E.9 (Base for a Topology). Let $\mathcal{T}$ be a topology on a set X. A *base* for the topology is a collection of open sets $\mathcal{B} \subseteq \mathcal{T}$ such that for any open set $U \in \mathcal{T}$ and any vector $x \in U$ there exists a base element $B \in \mathcal{B}$ that contains x and is contained in U, i.e.,

$$\forall\, U \in \mathcal{T}, \quad \forall\, x \in U, \quad \exists\, B \in \mathcal{B} \;\text{ such that }\; x \in B \subseteq U.$$

Consequently, if $\mathcal{B}$ is a base for a topology, then a set U is open if and only if

$$U \;=\; \bigcup_{x \in U} \bigl\{B_x : B_x \in \mathcal{B} \text{ and } x \in B_x \subseteq U\bigr\}.$$

In particular, if X is a metric space then the collection of open balls in X forms a base for the topology induced by the metric.

Remark E.10. If $\mathcal{E}$ is an arbitrary collection of subsets of X, then the smallest topology that contains $\mathcal{E}$ is called the *topology generated by* $\mathcal{E}$, denoted $\mathcal{T}(\mathcal{E})$, see Exercise A.38. Each element of $\mathcal{T}(\mathcal{E})$ can be written as a union of finite intersections of elements of $\mathcal{E}$. In contrast, if $\mathcal{B}$ is a base for a topology $\mathcal{T}$, then each element of $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$. The topology generated by $\mathcal{B}$ is $\mathcal{T}$, but we do not need to go through the extra step of taking finite intersections to form arbitrary open sets in the topology. Sometimes $\mathcal{E}$ is called a *subbase* for the topology $\mathcal{T}(\mathcal{E})$.

Definition E.11 (Locally Convex). If X is a vector space that has a topology $\mathcal{T}$, then we say that this topology is *locally convex* if there exists a base $\mathcal{B}$ for the topology that consists of convex sets.

For example, if X is a normed linear space, then X is locally convex since each open ball in X is convex. However, as $L^p(E)$ with $p < 1$ illustrates, a metric linear space need not be locally convex in general.

E.2.2 Topological Vector Spaces

A topological vector space is a vector space that has a topology such that the operations of vector addition and scalar multiplication are continuous. In order to define this precisely, the reader should recall the definition of the topology on a product space $X \times Y$ given in Section A.6.

Definition E.12 (Topological Vector Space). A *topological vector space* (TVS) is a vector space X together with a topology $\mathcal{T}$ such that

(a) $(x,y) \mapsto x+y$ is a continuous map of $X \times X$ into X, and

(b) $(c,x) \mapsto cx$ is a continuous map of $\mathbb{C} \times X$ into X.

By Exercise A.8, every normed linear space is a locally convex topological vector space.

Remark E.13. Some authors additionally require in the definition of topological vector space that the topology on X be Hausdorff, and some further require the topology to be locally convex.

Lemma E.14. *If X is a topological vector space, then the topology on X is* translation-invariant*, meaning that if $U \subseteq X$ is open, then $U + x$ is open for every $x \in X$.*

Proof. Suppose $U \subseteq X$ is open. Then, since vector addition is continuous, the inverse image of U under vector addition, which is

$$+^{-1}(U) \;=\; \big\{(y,z) \in X \times X : y+z \in U\big\},$$

is open in $X \times X$. Exercise A.41 therefore implies that the restriction

$$\big\{y \in X : y+z \in U\big\}$$

is open in X for each $z \in X$. In particular,

$$U + x \;=\; \big\{u + x : u \in U\big\} \;=\; \big\{y \in X : y + (-x) \in U\big\}$$

is open in X. □

Additional Problems

E.2. Let X be a normed vector space. Show that the topology induced from the norm is the smallest topology with respect to which X is a topological vector space and $x \mapsto \|x\|$ is continuous.

E.3. Let X be a normed vector space. For each $0 < r < s < \infty$, let A_{rs} be the open annulus $A_{rs} = \{x \in X : r < \|x\| < s\}$ centered at the origin. Let $\mathcal{B}$ consist of $\emptyset$, X, all open balls $B_r(0)$ centered at the origin, and all open annuli A_{rs} centered at the origin. Prove the following facts.

(a) $\mathcal{B}$ is a base for the topology $\mathcal{T}(\mathcal{B})$ generated by $\mathcal{B}$.

(b) $x \mapsto \|x\|$ is continuous with respect to the topology $\mathcal{T}(\mathcal{B})$.

(c) X is not a topological vector space with respect to the topology $\mathcal{T}(\mathcal{B})$.

E.3 Topologies Induced by Families of Seminorms

Our goal in this section is to show that a family of seminorms on a vector space X induces a natural topology on that space, and that X is a locally convex topological vector space with respect to that topology.

E.3.1 Motivation

In order to motivate the construction of the topology associated with a family of seminorms, let us consider the ordinary topology on the Euclidean space $\mathbb{R}^2$. We usually consider this topology to be induced from the Euclidean norm on $\mathbb{R}^2$. We will show that the same topology is induced from the two seminorms ρ_1 and ρ_2 on $\mathbb{R}^2$ defined by

$$\rho_1(x_1, x_2) = |x_1| \qquad \text{and} \qquad \rho_2(x_1, x_2) = |x_2|.$$

In analogy with how we create open balls from a norm, define

$$B_r^\alpha(x) \;=\; \{y \in \mathbb{R}^2 : \rho_\alpha(x-y) < r\}, \qquad x \in \mathbb{R}^2,\ r > 0,\ \alpha = 1, 2.$$

These sets are "open strips" instead of open balls, see the illustration in Figure E.2. By taking finite intersections of these strips, we obtain all possible open rectangles $(a, b) \times (c, d)$, and unions of these rectangles exactly give us all the subsets of $\mathbb{R}^2$ that are open with respect to the Euclidean topology. Thus

$$\mathcal{E} \;=\; \bigl\{B_r^\alpha(x) : x \in \mathbb{R}^2,\ r > 0,\ \alpha = 1, 2\bigr\}$$

generates the usual topology on $\mathbb{R}^2$. However, $\mathcal{E}$ is not a base for this topology, since we cannot write an arbitrary open set as a union of open strips. Instead, we have to take one more step: The collection of *finite intersections* of the open strips forms the base. Every open set is a union of finite intersections of open strips. Furthermore, each of these finite intersections of strips is an open rectangle, which is convex, so our base consists of convex sets. Thus, this topology is locally convex. The topology induced from an arbitrary family of seminorms on a vector space will be defined in exactly the same way.

E.3.2 The Topology Associated with a Family of Seminorms

Definition E.15 (Topology Induced from Seminorms). Let $\{\rho_\alpha\}_{\alpha \in J}$ be a family of seminorms on a vector space X. Then the *αth open strip of radius r centered at $x \in X$* is

$$B_r^\alpha(x) \;=\; \bigl\{y \in X : \rho_\alpha(x-y) < r\bigr\}.$$

Let $\mathcal{E}$ be the collection of all open strips in X:

$$\mathcal{E} \;=\; \bigl\{B_r^\alpha(x) : \alpha \in J,\ r > 0,\ x \in X\bigr\}.$$

The topology $\mathcal{T}(\mathcal{E})$ generated by $\mathcal{E}$ is called the *topology induced by* $\{\rho_\alpha\}_{\alpha \in J}$.

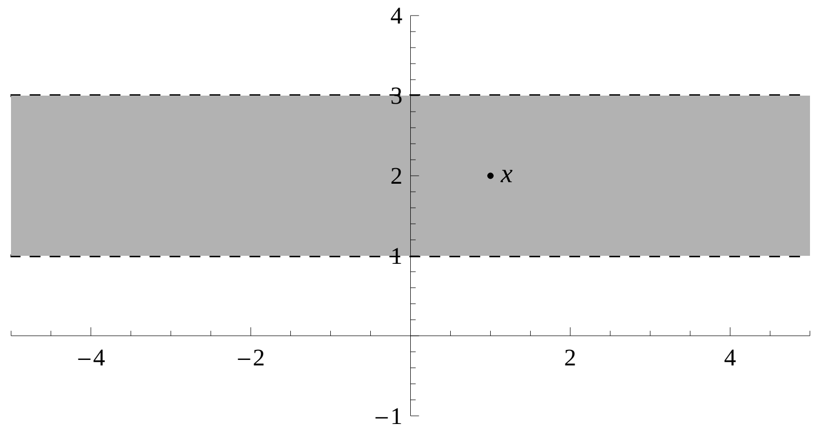

Fig. E.2. The open strip $B_r^2(x)$ for $x = (1, 2)$ and $r = 1$.

The fact that ρ_α is a seminorm ensures that each open strip $B_r^\alpha(x)$ is convex. Hence all finite intersections of open strips will also be convex.

One base for the topology generated by the open strips is the collection of all possible finite intersections of open strips. However, it is usually more notationally convenient to use the somewhat smaller base consisting of finite intersections of strips that are all centered at the same point and have the same radius.

Theorem E.16. *Let $\{\rho_\alpha\}_{\alpha \in J}$ be a family of seminorms on a vector space X. Then*

$$\mathcal{B} = \left\{ \bigcap_{j=1}^{n} B_r^{\alpha_j}(x) : n \in \mathbb{N},\ \alpha_j \in J,\ r > 0,\ x \in X \right\}$$

forms a base for the topology induced from these seminorms. In fact, if U is open and $x \in U$, then there exists an $r > 0$ and $\alpha_1, \dots, \alpha_n \in J$ such that

$$\bigcap_{j=1}^{n} B_r^{\alpha_j}(x) \subseteq U.$$

Further, every element of $\mathcal{B}$ is convex.

Proof. Suppose that $U \subseteq X$ is open and $x \in U$. By the characterization of the generated topology given in Exercise A.38, U is a union of finite intersections of elements of $\mathcal{E}$. Hence we have

$$x \in \bigcap_{j=1}^{n} B_{r_j}^{\alpha_j}(x_j)$$

for some $n > 0$, $\alpha_j \in J$, $r_j > 0$, and $x_j \in X$. Since $x \in B_{r_j}^{\alpha_j}(x_j)$, we have $\rho_{\alpha_j}(x - x_j) < r_j$ for each j. Therefore, if we set

$$r = \min\{r_j - \rho_{\alpha_j}(x - x_j) : j = 1, \dots, n\},$$

then we have $B_r^{\alpha_j}(x) \subseteq B_{r_j}^{\alpha_j}(x_j)$ for each $j = 1, \dots, n$. Hence

$$B = \bigcap_{j=1}^{n} B_r^{\alpha_j}(x) \in \mathcal{B},$$

and we have $x \in B \subseteq U$. □

Note that even if there are infinitely many seminorms in our family, when constructing the base $\mathcal{B}$ we only intersect finitely many strips at a time.

Fortunately, the topology induced by a family of seminorms is always locally convex. Unfortunately, it need not be Hausdorff in general.

Exercise E.17. Let $\{\rho_\alpha\}_{\alpha \in J}$ be a family of seminorms on a vector space X. Show that the induced topology on X is Hausdorff if and only if

$$\rho_\alpha(x) = 0 \text{ for all } \alpha \in J \quad \Longleftrightarrow \quad x = 0.$$

Hausdorffness is an important property because without it the limit of a sequence need not be unique (see Problem A.17). Hence in almost every practical circumstance we require the topology to be Hausdorff. If any one of the seminorms in our family is a norm, then the corresponding topology is automatically Hausdorff (for example, this is the case for $C_b^\infty(\mathbb{R})$, see Example E.3). On the other hand, the topology can be Hausdorff even if no individual seminorm is a norm (consider $L^1_{\mathrm{loc}}(\mathbb{R})$ in Example E.5).

E.3.3 The Convergence Criterion

The meaning of convergence with respect to a net in an arbitrary topological space X was given in Definition A.44. Specifically, a net $\{x_i\}_{i \in I}$ converges to $x \in X$ if for any open neighborhood U of x there exists $i_0 \in I$ such that

$$i \geq i_0 \implies x_i \in U.$$

In this case we write $x_i \to x$.

When the topology is induced from a family of seminorms, we can reformulate the meaning of convergence directly in terms of the seminorms instead of open neighborhoods. Since we are dealing with arbitrary collections of seminorms at this point, we must still deal with convergence in terms of nets rather than ordinary sequences, but even so the fact that seminorms are real-valued allows a certain amount of notational simplication. Specifically, given a net $\{x_i\}_{i \in I}$ in X and given a seminorm ρ on X, since the open intervals form a base for the topology on $\mathbb{R}$, we have that $\rho(x_i) \to 0$ in $\mathbb{R}$ with respect to the directed set I if and only if for every $\varepsilon > 0$ there exists an $i_0 \in I$ such that

$$i \geq i_0 \implies \rho(x_i) < \varepsilon.$$

The next theorem shows that convergence with respect to the topology induced from a family of seminorms is exactly what we expect it should be, namely, simultaneous convergence with respect to each individual seminorm.

Theorem E.18. *Let X be a vector space whose topology is induced from a family of seminorms $\{\rho_\alpha\}_{\alpha\in J}$. Then given any net $\{x_i\}_{i\in I}$ and any $x \in X$, we have*

$$x_i \to x \quad\iff\quad \forall\, \alpha \in J,\ \rho_\alpha(x - x_i) \to 0.$$

Proof. $\Rightarrow$. Suppose that $x_i \to x$, and fix any $\alpha \in J$ and $\varepsilon > 0$. Then $B^\alpha_\varepsilon(x)$ is an open neighborhood of x, so by definition of convergence with respect to a net, there exists an $i_0 \in I$ such that

$$i \ge i_0 \quad\Longrightarrow\quad x_i \in B^\alpha_\varepsilon(x).$$

Therefore, for all $i \ge i_0$ we have $\rho_\alpha(x - x_i) < \varepsilon$, so $\rho_\alpha(x - x_i) \to 0$.

$\Leftarrow$. Suppose that $\rho_\alpha(x - x_i) \to 0$ for every $\alpha \in J$, and let U be any open neighborhood of x. Then by Theorem E.16, we can find an $r > 0$ and finitely many $\alpha_1, \dots, \alpha_n \in J$ such that

$$x \;\in\; \bigcap_{j=1}^{n} B^{\alpha_j}_r(x) \;\subseteq\; U.$$

Now, given any $j = 1, \dots, n$ we have $\rho_{\alpha_j}(x - x_i) \to 0$. Hence, for each j we can find a $k_j \in I$ such that

$$i \ge k_j \quad\Longrightarrow\quad \rho_{\alpha_j}(x - x_i) < r.$$

Since I is a directed set, there exists some $i_0 \in I$ such that $i_0 \ge k_j$ for $j = 1, \dots, n$. Thus, for all $i \ge i_0$ we have $\rho_{\alpha_j}(x - x_i) < r$ for each $j = 1, \dots, n$, so

$$x_i \;\in\; \bigcap_{j=1}^{n} B^{\alpha_j}_r(x) \;\subseteq\; U, \qquad i \ge i_0.$$

Hence $x_i \to x$. □

By combining Theorem E.18 with Lemma A.53, we obtain a criterion for continuity in terms of the seminorms.

Corollary E.19. *Let X be a vector space whose topology is induced from a family of seminorms $\{\rho_\alpha\}_{\alpha\in J}$, let Y be any topological space, and fix $x \in X$. Then the following two statements are equivalent.*

(a) *$f\colon X \to Y$ is continuous at x.*

(b) *For any net $\{x_i\}_{i\in I}$ in X,*

$$\rho_\alpha(x - x_i) \to 0 \ \text{ for each } \alpha \in J \quad\Longrightarrow\quad f(x_i) \to f(x) \text{ in } Y.$$

In particular, if $\mu\colon X \to \mathbb{C}$ is a linear functional, then μ is continuous if and only if for each net $\{x_i\}_{i\in I}$ in X we have

$$\rho_\alpha(x_i) \to 0 \ \text{ for each } \alpha \in J \quad\Longrightarrow\quad \langle x_i, \mu\rangle \to 0.$$

The *dual space* X^* of X is the space of all continuous linear functionals on X.

Remark E.20. Because of the Reverse Triangle Inequality, $\rho_\alpha(x - x_i) \to 0$ implies $\rho_\alpha(x_i) \to \rho_\alpha(x)$. Hence each seminorm ρ_α is continuous with respect to the induced topology.

E.3.4 Continuity of the Vector Space Operations

Now we can show that a vector space with a topology induced from a family of seminorms is a topological vector space.

Theorem E.21. *If X is a vector space whose topology is induced from a family of seminorms $\{\rho_\alpha\}_{\alpha\in J}$, then X is a locally convex topological vector space.*

Proof. We have already seen that there is a base for the topology that consists of convex open sets, so we just have to show that vector addition and scalar multiplication are continuous with respect to this topology.

Suppose that $\{(c_i, x_i)\}_{i\in I}$ is any net in $\mathbb{C} \times X$, and that $(c_i, x_i) \to (c, x)$ with respect to the product topology on $\mathbb{C} \times X$. By Problem A.21, this is equivalent to assuming that $c_i \to c$ in $\mathbb{C}$ and $x_i \to x$ in X. Fix any $\alpha \in J$ and any $\varepsilon > 0$. Suppose that $\rho_\alpha(x) \neq 0$. Since $\rho_\alpha(x - x_i) \to 0$, there exist i_1, $i_2 \in I$ such that

$$i \geq i_1 \quad\implies\quad |c - c_i| < \min\left\{\frac{\varepsilon}{2\,\rho_\alpha(x)},\ 1\right\}.$$

and

$$i \geq i_2 \quad\implies\quad \rho_\alpha(x - x_i) < \frac{\varepsilon}{2\,(|c|+1)}$$

By definition of directed set, there exists some $i_0 \geq i_1, i_2$, so both of these inequalities hold for $i \geq i_0$. In particular, $\{c_i\}_{i\geq i_0}$ is a bounded sequence, with $|c_i| < |c| + 1$ for all $i \geq i_0$. Hence, for $i \geq i_0$ we have

$$\begin{aligned}
\rho_\alpha(cx - c_i x_i) &\leq \rho_\alpha(cx - c_i x) + \rho_\alpha(c_i x - c_i x_i) \\
&= |c - c_i|\,\rho_\alpha(x) + |c_i|\,\rho_\alpha(x - x_i) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

If $\rho_\alpha(x) = 0$ then we similarly obtain $\rho_\alpha(cx - c_i x_i) < \varepsilon/2$ for $i \geq i_0$. Thus we have $\rho_\alpha(cx - c_i x_i) \to 0$. Since this is true for every α, Corollary E.19 implies that $c_i x_i \to cx$.

Exercise: Show that vector addition is continuous. □

E.3.5 Continuity Equals Boundedness

For linear maps on normed vector spaces, Theorem C.6 tells us that continuity is equivalent to boundedness. We will now prove an analogous result for operators on vector spaces whose topogies are induced from families of seminorms. In the statement of the following result, it is perhaps surprising at first glance that "boundedness" of a given operator is completely determined by a fixed *finite* subcollection of the seminorms. This is a reflection of the construction of the topology, and specifically of the fact that a base for the topology is obtained by intersecting only finitely many open strips at a time.

Theorem E.22 (Continuity Equals Boundedness). *Let X be a vector space whose topology is induced from a family of seminorms $\{\rho_\alpha\}_{\alpha \in I}$. Let Y be a vector space whose topology is induced from a family of seminorms $\{q_\beta\}_{\beta \in J}$. If $L\colon X \to Y$ is linear, then the following statements are equivalent.*

(a) *L is continuous.*

(b) *For each $\beta \in J$, there exist $N \in \mathbb{N}$, $\alpha_1, \dots, \alpha_N \in I$, and $C > 0$ (all depending on β) such that*

$$q_\beta(Lx) \;\le\; C \sum_{j=1}^{N} \rho_{\alpha_j}(x), \qquad x \in X. \tag{E.1}$$

Proof. (a) $\Rightarrow$ (b). Assume that L is continuous, and fix $\beta \in J$. Since q_β is continuous, so is $q_\beta \circ L$. Hence

$$(q_\beta \circ L)^{-1}(-1,1) \;=\; \{x \in X : q_\beta(Lx) < 1\}$$

is open in X. Further, this set contains $x = 0$, so there must exist a base set $B = \cap_{j=1}^{N} B_r^{\alpha_j}(0)$ such that

$$B \;\subseteq\; \{x \in X : q_\beta(Lx) < 1\}. \tag{E.2}$$

We will show that equation (E.1) is satisfied with $C = 2/r$. To see this, fix any $x \in X$, and set

$$\delta \;=\; \sum_{j=1}^{N} \rho_{\alpha_j}(x).$$

Case 1: $\delta = 0$. In this case, given any $\lambda > 0$ we have

$$\rho_{\alpha_j}(\lambda x) \;=\; \lambda\, \rho_{\alpha_j}(x) \;=\; 0, \qquad j = 1, \dots, N.$$

Hence $\lambda x \in B_r^{\alpha_j}(0)$ for each $j = 1, \dots, N$, so $\lambda x \in B$. In light of the inclusion in equation (E.2), we therefore have for every $\lambda > 0$ that

$$\lambda\, q_\beta(Lx) \;=\; q_\beta(L(\lambda x)) \;<\; 1.$$

Therefore $q_\beta(Lx) = 0$. Hence in this case the inequality (E.1) is trivially satisfied.

Case 2: $\delta > 0$. In this case we have for each $j = 1, \dots, N$ that

$$\rho_{\alpha_j}\Big(\frac{rx}{2\delta}\Big) \;=\; \frac{r}{2\delta}\, \rho_{\alpha_j}(x) \;\le\; \frac{r}{2} \;<\; r,$$

so $\frac{rx}{2\delta} \in B$. Considering equation (E.2), we therefore have

$$\frac{r}{2\delta}\, q_\beta(Lx) \;=\; q_\beta\Big(L\Big(\frac{rx}{2\delta}\Big)\Big) \;<\; 1.$$

Hence

$$q_\beta(Lx) \;<\; \frac{2\delta}{r} \;=\; \frac{2}{r}\sum_{j=1}^{N} \rho_{\alpha_j}(x),$$

which is the desired inequality.

(b) ⇒ (a). Suppose that statement (b) holds. Let $\{x_i\}_{i\in A}$ be any net in X, and suppose that $x_i \to x$ in X. Then for each $\alpha \in I$, we have by Corollary E.19 that $\rho_\alpha(x - x_i) \to 0$ for each $\alpha \in I$. Equation (E.1) therefore implies that $q_\beta(Lx - Lx_i) \to 0$ for each $\beta \in J$. Using Corollary E.19 again, this says that $Lx_i \to Lx$ in Y, so L is continuous. □

Often, the space Y will be a normed space. In this case, the statement of Theorem E.22 simplifies as follows.

Corollary E.23. *Let X be a vector space whose topology is induced from a family of seminorms $\{\rho_\alpha\}_{\alpha\in J}$, and let Y be a normed linear space. If $L\colon X \to Y$ is linear, then the following statements are equivalent.*

(a) *L is continuous.*

(b) *There exist $N \in \mathbb{N}$, $\alpha_1, \dots, \alpha_N \in J$, and $C > 0$ such that*

$$\|Lx\| \;\le\; C\sum_{j=1}^{N} \rho_{\alpha_j}(x), \qquad x \in X. \tag{E.3}$$

Thus, to show that a given linear operator $L\colon X \to Y$ is continuous, we need only find *finitely many* seminorms such that the boundedness condition in equation (E.3) holds.

E.4 Topologies Induced by Countable Families of Seminorms

In this section, we will show that if a Hausdorff topology is induced from a *countable* collection of seminorms, then it is metrizable. In particular, we can define continuity of operators on such a space using convergence of ordinary sequences instead of nets.

E.4.1 Metrizing the Topology

Exercise E.24. Let X be a vector space whose topology is induced from a countable family of seminorms $\{\rho_n\}_{n\in\mathbb{N}}$, and assume that X is Hausdorff. Prove the following statements.

(a) The function

$$\mathrm{d}(x,y) \;=\; \sum_{n=1}^{\infty} 2^{-n}\,\frac{\rho_n(x-y)}{1+\rho_n(x-y)}$$

defines a metric on X.

(b) The metric d generates the same topology as the family of seminorms $\{\rho_n\}_{n\in\mathbb{N}}$.
(c) The metric d is translation-invariant, i.e.,

$$\mathrm{d}(x+z, y+z) \;=\; \mathrm{d}(x,y), \qquad x,y,z \in X.$$

Since the metric defines the same topology as the family of seminorms, they define the same convergence criteria, i.e., given a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X and given $x \in X$, we have

$$\mathrm{d}(x, x_k) \to 0 \quad\iff\quad \forall\, n \in \mathbb{N},\ \rho_n(x - x_k) \to 0.$$

Definition E.25 (Fréchet Space). Let X be a Hausdorff vector space whose topology is induced from a countable family of seminorms $\{\rho_n\}_{n\in\mathbb{N}}$. If X is complete with respect to the metric constructed in Exercise E.24, then we call X a *Fréchet space*.

Exercise E.26. Let X be as above, and suppose that a sequence $\{x_k\}_{k\in\mathbb{N}}$ is Cauchy with respect to the metric d. Show that $\{x_k\}_{k\in\mathbb{N}}$ is Cauchy with respect to each individual seminorm ρ_n.

Exercise E.27. Show that the spaces $C_b^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, $L^1_{\mathrm{loc}}(\mathbb{R})$, and $C^\infty(\mathbb{R})$ considered in Examples E.3–E.6 are all Fréchet spaces.

The topology on a Fréchet space need not be normable. For example, we will show that this is the case for the space $C^\infty(\mathbb{R})$.

Example E.28. The topology on $C^\infty(\mathbb{R})$ is induced from the family of seminorms $\{\rho_{mn}\}_{m,n\ge 0}$, where $\rho_{mn}(f) = \|f^{(n)}\cdot\chi_{[-m,m]}\|_\infty$ for $f \in C^\infty(\mathbb{R})$. Suppose that there was some norm $\|\cdot\|$ that induced this topology, and let $U = \{f \in C^\infty(\mathbb{R}) : \|f\| < 1\}$ be the unit open ball. Then U must also be open with respect to the topology induced from the seminorms, so there exists some base element

$$\begin{aligned} B \;&=\; \bigcap_{j=1}^{k} B_r^{m_k,n_k}(0) \\ &=\; \bigl\{f \in C^\infty(\mathbb{R}) : \|f^{(n_j)}\cdot\chi_{[-m_j,m_j]}\|_\infty < r \text{ for } j = 1,\dots,k\bigr\} \end{aligned}$$

that is contained in U. Let $M = \max\{m_1,\dots,m_k\}$, and let f be any nonzero function in $C^\infty(\mathbb{R})$ that vanishes on $[-M,M]$. Then $cf \in B \subseteq U$ for every $c \in \mathbb{R}$, which is a contradiction since $\|f\| \ne 0$.

We remark that many of the "big" theorems of functional analysis that we stated in Appendix C have extensions to the setting of Fréchet spaces (and beyond). The essence of the Hahn–Banach Theorem is convexity, while the essence of the Baire Category Theorem, Open Mapping Theorem, and Closed Graph Theorems is completeness. As Fréchet spaces are both locally convex and complete, it is not too surprising that these theorems can be extended to Frećhet spaces. In particular, Problem 4.43 in the main part of this volume requires an application of the Closed Graph Theorem to the Fréchet space $L^1_{\text{loc}}(\mathbb{R})$. Therefore we state a particular extension of the Closed Graph Theorem here. This is a special case of [Rud91, Thm. 2.15].

Theorem E.29 (Closed Graph Theorem for Fréchet Spaces). *Let X and Y be Fréchet spaces. If $A\colon X \to Y$ is linear, then the following statements are equivalent.*

(a) *A is continuous.*

(b) $\operatorname{graph}(A) = \big\{(f, Af) : f \in X\big\}$ *is a closed subset of $X \times Y$.*

(c) *If $f_n \to f$ in X and $Af_n \to g$ in Y, then $g = Af$.*

E.4.2 Tempered and Compactly Supported Distributions

The *dual space* X^* of a topological vector space X is the space of all continuous linear functionals on X. Restating our earlier "continuity equals boundedness" results for the specific case of linear functionals gives us the following result.

Theorem E.30. *Let X be a vector space whose topology is induced from a family of seminorms $\{\rho_\alpha\}_{\alpha \in J}$, and let $\mu\colon X \to \mathbb{C}$ be linear. Then the following statements are equivalent.*

(a) *μ is continuous, i.e., $\mu \in X^*$.*

(b) *There exist $N \in \mathbb{N}$, $\alpha_1, \dots, \alpha_N \in J$, and $C > 0$ such that*

$$|\langle x, \mu\rangle| \;\le\; C \sum_{j=1}^{N} \rho_{\alpha_j}(x), \qquad x \in X. \tag{E.4}$$

We will apply this to two particular dual spaces, which are called the space of tempered distributions and the space of compactly supported distributions. Each of these is studied in more detail in Chapter 3. Both are subsets of the space of distributions, which is defined in Section E.5 and is also studied in detail in Chapter 3.

Definition E.31 (Tempered Distributions). The space of *tempered distributions* $\mathcal{S}'(\mathbb{R})$ is the dual space of $\mathcal{S}(\mathbb{R})$:

$$\mathcal{S}'(\mathbb{R}) \;=\; \mathcal{S}(\mathbb{R})^* \;=\; \big\{\mu\colon \mathcal{S}(\mathbb{R}) \to \mathbb{C} \,:\, \mu \text{ is linear and continuous}\big\}.$$

Since the topology on $\mathcal{S}(\mathbb{R})$ is induced from the countably many seminorms

$$\rho_{mn}(f) \;=\; \|x^m f^{(n)}(x)\|_\infty, \qquad m, n \ge 0,$$

it is metrizable. Hence we can formulate continuity of operators on the Schwartz space in terms of convergence of ordinary sequences. Combining this with Theorem E.30 gives us the following characterization of bounded linear functionals on the Schwartz space.

Theorem E.32. *If* $\mu\colon \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ *is linear, then the following statements are equivalent.*

(a) μ *is continuous, i.e.,* $\mu \in \mathcal{S}'(\mathbb{R})$.

(b) *If* $f_k \in \mathcal{S}(\mathbb{R})$ *and* $f_k \to 0$ *in* $\mathcal{S}(\mathbb{R})$, *then* $\langle f_k, \mu\rangle \to 0$.

(c) *There exist* $C > 0$ *and* $M,\, N \geq 0$ *such that*

$$|\langle f, \mu\rangle| \;\leq\; C \sum_{m=0}^{M} \sum_{n=0}^{N} \|x^m f^{(n)}(x)\|_\infty, \qquad f \in \mathcal{S}(\mathbb{R}).$$

Example E.33. Define $\delta\colon \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ by $\langle f, \delta\rangle = f(0)$. Given any $f \in \mathcal{S}(\mathbb{R})$, we have

$$|\langle f, \delta\rangle| \;=\; |f(0)| \;\leq\; \|f\|_\infty \;=\; \rho_{00}(f).$$

Theorem E.32 therefore implies that δ is continuous, and δ is an example of a tempered distribution. For this particular operator, "boundedness" is satisfied using an estimate involving only the single seminorm ρ_{00}.

Similarly, the functional $\delta'\colon \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ given by $\langle f, \delta'\rangle = -f'(0)$ is continuous because

$$|\langle f, \delta'\rangle| \;=\; |f'(0)| \;\leq\; \|f'\|_\infty \;=\; \rho_{01}(f),$$

and we can likewise define higher derivatives of δ by setting

$$\langle f, \delta^{(j)}\rangle \;=\; (-1)^j\, f^{(j)}(0). \tag{E.5}$$

In this sense, the delta functional is "infinitely differentiable" in $\mathcal{S}'(\mathbb{R})$. The exact meaning of derivatives of distributions is discussed in Section 4.6.

Next we consider the dual space of $\mathcal{E}(\mathbb{R}) = C^\infty(\mathbb{R})$.

Definition E.34 (Compactly Supported Distributions). The space of *compactly supported distributions* $\mathcal{E}'(\mathbb{R})$ is the dual space of $C^\infty(\mathbb{R}) = \mathcal{E}(\mathbb{R})$:

$$\mathcal{E}'(\mathbb{R}) \;=\; C^\infty(\mathbb{R})^* \;=\; \bigl\{\mu\colon C^\infty(\mathbb{R}) \to \mathbb{C} : \mu \text{ is linear and continuous}\bigr\}.$$

Since the topology on $C^\infty(\mathbb{R})$ is induced from the countably many seminorms $\rho_{m,n}(f) = \|f^{(n)} \cdot \chi_{[-m,m]}\|_\infty$ with $m,\, n \geq 0$, it is metrizable. Hence we obtain the following characterization of bounded linear functionals on $C^\infty(\mathbb{R})$.

Theorem E.35. *If* $\mu\colon C^\infty(\mathbb{R}) \to \mathbb{C}$ *is linear, then the following statements are equivalent.*

(a) μ *is continuous, i.e.,* $\mu \in \mathcal{E}'(\mathbb{R})$.

(b) *If* $f_k \in C^\infty(\mathbb{R})$ *and* $f_k \to 0$ *in* $C^\infty(\mathbb{R})$, *then* $\langle f_k, \mu\rangle \to 0$.

(c) *There exist* $C > 0$ *and* $M,\, N \geq 0$ *such that*

$$|\langle f, \mu\rangle| \;\leq\; C \sum_{n=0}^{N} \|f^{(n)} \cdot \chi_{[-M,M]}\|_\infty, \qquad f \in C^\infty(\mathbb{R}).$$

The following exercise explains why the terminology "compactly supported" is used in connection with the dual space of $C^\infty(\mathbb{R})$.

Exercise E.36. Show that if $\mu \in \mathcal{E}'(\mathbb{R})$, then there exists a compact set $K \subseteq \mathbb{R}$ such that if $f \in C^\infty(\mathbb{R})$ and $\operatorname{supp}(f) \subseteq \mathbb{R}\backslash K$, then $\langle f, \mu \rangle = 0$.

The *support* of μ is defined precisely in Section 4.5. For example, $\delta^{(j)}$ as defined in equation (E.5) belongs to $\mathcal{E}'(\mathbb{R})$ and its support is exactly $\operatorname{supp}(\delta^{(n)}) = \{0\}$ (see Exercises 4.11 and 4.52).

E.5 $C_c^\infty(\mathbb{R})$ and its Dual Space $\mathcal{D}'(\mathbb{R})$

Not every topological vector space has a topology easily described by an explicit family of seminorms. The most important such space for us is

$$\mathcal{D}(\mathbb{R}) \;=\; C_c^\infty(\mathbb{R}) \;=\; \bigl\{f \in C^\infty(\mathbb{R}) : \operatorname{supp}(f) \text{ is compact}\bigr\}.$$

We will sketch some facts about the topology on $C_c^\infty(\mathbb{R})$ in this section, but will not prove every detail. For these, we refer the reader to functional analysis text such as Conway [Con90] or Rudin [Rud91].

E.5.1 The Topology on $C_c^\infty(\mathbb{R})$

Definition E.37. Given a compact set $K \subseteq \mathbb{R}$, we let $C^\infty(K)$ denote the collection of infinitely differentiable functions on the real line that are supported *within* K:

$$C^\infty(K) \;=\; \bigl\{f \in C^\infty(\mathbb{R}) : \operatorname{supp}(f) \subseteq K\bigr\}.$$

With K fixed, $C^\infty(K)$ has a topology defined by the countable family of seminorms $\{\rho_{K,n}\}_{n\ge 0}$, where

$$\rho_{K,n}(f) \;=\; \|f^{(n)} \cdot \chi_K\|_\infty \;=\; \|f^{(n)}\|_\infty, \qquad f \in C^\infty(K). \tag{E.6}$$

Further, $C^\infty(K)$ is complete with respect to this topology, so it is a Fréchet space.

As a set, we have

$$C_c^\infty(\mathbb{R}) \;=\; \bigcup \bigl\{C^\infty(K) : K \subseteq \mathbb{R},\, K \text{ compact}\bigr\}.$$

However, we do not define the topology on $C_c^\infty(\mathbb{R})$ by simply combining all of the seminorms $\rho_{K,n}$ for each $n \ge 0$ and compact set K. Indeed, as we saw in Example E.6, $\{\rho_{K,n} : n \ge 0, \text{ compact } K \subseteq \mathbb{R}\}$ is exactly the family of seminorms that induces the topology on $C^\infty(\mathbb{R})$. Since $C_c^\infty(\mathbb{R})$ is contained in $C^\infty(\mathbb{R})$, this family of seminorms does induce a topology on $C_c^\infty(\mathbb{R})$, but it is not the "correct" topology for $C_c^\infty(\mathbb{R})$. In particular, while $C^\infty(\mathbb{R})$ is complete in this topology, $C_c^\infty(\mathbb{R})$ is not (see Problem E.4). Instead, the "appropriate" topology on $C_c^\infty(\mathbb{R})$ is obtained by forming the *inductive limit* of the topologies on $C^\infty([-N,N])$ for $N \in \mathbb{N}$. To motivate this, set $K_N = [-N,N]$, and note the following facts.

(a) $C_c^\infty(\mathbb{R})$ is a vector space.

(b) $C^\infty(K_N)$ is a vector space whose topology is defined by a family of seminorms, and this topology is Hausdorff.

(c) $\mathbb{N}$ is a directed set, and if $M \le N$ then $C^\infty(K_M) \subseteq C^\infty(K_N)$.

(d) If $M \le N$ and U is an open subset of $C^\infty(K_N)$, then $U \cap C^\infty(K_M)$ is an open subset of $C^\infty(K_M)$. That is, the embedding $C^\infty(K_M) \hookrightarrow C^\infty(K_N)$ is continuous.

(e) $C_c^\infty(\mathbb{R}) = \bigcup \bigl\{C^\infty(K_N) : N \in \mathbb{N}\bigr\}$.

In the terminology of [Con90], $\bigl(C_c^\infty(\mathbb{R}),\, \{C^\infty(K_N)\}_{N\in\mathbb{N}}\bigr)$ is an *inductive system*, and $C_c^\infty(\mathbb{R})$ inherits its topology from this system. We will refer to this topology as the *inductive limit topology* on $C_c^\infty(\mathbb{R})$. A similar topology on the space $C_c(\mathbb{R})$ was discussed in Section D.8.

To give the precise definition of this topology, let us say that a subset $V \subseteq C_c^\infty(\mathbb{R})$ is *balanced* if

$$f \in V,\ |\alpha| < 1 \implies \alpha f \in V.$$

Define $\mathcal{B}$ to be the collection of all $V \subseteq C_c^\infty(\mathbb{R})$ such that V is convex and balanced, and $V \cap C^\infty(K_N)$ is open in $C^\infty(K_N)$ for each N. Then a set $U \subseteq C_c^\infty(\mathbb{R})$ is defined to be open if for each $f \in U$, there exists a set $V \in \mathcal{B}$ such that $f + V \subseteq U$. With this topology, $C_c^\infty(\mathbb{R})$ is a locally convex topological vector space. There does exist a family of seminorms that induces this topology, but it is not a metrizable topology.

This explicit definition of the topology is not very revealing. For us, the following theorem characterizing the convergence criterion in $C_c^\infty(\mathbb{R})$ is far more important. In effect, we take the next theorem as the definition of convergence in $C_c^\infty(\mathbb{R})$.

Theorem E.38 (Convergence in $C_c^\infty(\mathbb{R})$). *If $C_c^\infty(\mathbb{R})$ is given the inductive limit topology described above, then the following statements hold.*

(a) *A sequence $\{f_k\}_{k\in\mathbb{N}}$ is Cauchy in $C_c^\infty(\mathbb{R})$ if and only if there exists a compact set $K \subseteq \mathbb{R}$ such that $f_k \in C^\infty(K)$ for each k and $\{f_k\}_{k\in\mathbb{N}}$ is Cauchy in $C^\infty(K)$.*

(b) *A sequence $\{f_k\}_{k\in\mathbb{N}}$ converges to f in $C_c^\infty(\mathbb{R})$ if and only if there exists a compact set $K \subseteq \mathbb{R}$ such that $f_k \in C^\infty(K)$ for each k and $f_k \to f$ in $C^\infty(K)$.*

Since each space $C^\infty(K)$ is a Fréchet space, it follows that every Cauchy sequence in $C_c^\infty(\mathbb{R})$ converges in $C_c^\infty(\mathbb{R})$. Thus $C_c^\infty(\mathbb{R})$ is complete with respect to this topology.

It is useful to explicitly restate the convergence criteria in $C_c^\infty(\mathbb{R})$ as follows: A sequence $\{f_k\}_{k\in\mathbb{N}}$ converges to f in $C_c^\infty(\mathbb{R})$ if and only if there exists a single compact set $K \subseteq \mathbb{R}$ such that $\operatorname{supp}(f_k) \subseteq K$ for each k, and

$$\forall\, n \geq 0, \quad \lim_{k\to\infty} \|f^{(n)} - f_k^{(n)}\|_\infty \;=\; 0.$$

In particular, convergence in $C_c^\infty(\mathbb{R})$ requires that all the elements of the sequence be supported within one single compact set.

Continuity can then be defined in the usual way by using convergence of sequences, as in the next exercise.

Exercise E.39. Prove that differentiation is a continuous mapping of $C_c^\infty(\mathbb{R})$ into itself. That is, show that if $f_k \to f$ in $C_c^\infty(\mathbb{R})$, then $f_k' \to f'$ in $C_c^\infty(\mathbb{R})$.

E.5.2 The Space of Distributions

Definition E.40. A linear functional $\mu\colon C_c^\infty(\mathbb{R}) \to \mathbb{C}$ is continuous if

$$f_k \to 0 \;\text{ in } C_c^\infty(\mathbb{R}) \implies \langle f_k, \mu\rangle \to 0.$$

The space of *distributions* $\mathcal{D}'(\mathbb{R})$ is the dual space of $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$:

$$\mathcal{D}'(\mathbb{R}) \;=\; C_c^\infty(\mathbb{R})^* \;=\; \bigl\{\mu\colon C_c^\infty(\mathbb{R}) \to \mathbb{C} : \mu \text{ is linear and continuous}\bigr\}.$$

Combining Theorem E.38 with our previous "continuity equals boundedness" results (Corollary E.23) yields the following characterization of continuous linear functionals on $C_c^\infty(\mathbb{R})$.

Theorem E.41. *If $\mu\colon C_c^\infty(\mathbb{R}) \to \mathbb{C}$ is linear, then the following statements are equivalent.*

(a) *μ is continuous, i.e., $\mu \in \mathcal{D}'(\mathbb{R})$.*

(b) *$\mu|_{C^\infty(K)}$ is continuous for each compact $K \subseteq \mathbb{R}$. That is, if $K \subseteq \mathbb{R}$ is compact and $f_k \in C^\infty(K)$ for $k \in \mathbb{N}$, then*

$$f_k \to 0 \;\text{ in } C^\infty(K) \implies \langle f_k, \mu\rangle \to 0.$$

(c) *For each compact $K \subseteq \mathbb{R}$, there exist $C_K > 0$ and $N_K \geq 0$ such that*

$$|\langle f, \mu\rangle| \;\leq\; C_K \sum_{n=0}^{N_K} \|f^{(n)}\|_\infty, \qquad f \in C^\infty(K). \tag{E.7}$$

Proof. (a) $\Rightarrow$ (b). Suppose that $\mu \in \mathcal{D}'(\mathbb{R})$, and fix any compact $K \subseteq \mathbb{R}$. Suppose that $f_k \to 0$ in $C^\infty(K)$. Then, by definition, $f_k \to 0$ in $C_c^\infty(\mathbb{R})$, and therefore $\langle f_k, \mu\rangle \to 0$ since μ is continuous.

(b) $\Rightarrow$ (c). Suppose that statement (b) holds, and let $K \subseteq \mathbb{R}$ be compact. Then μ restricted to $C^\infty(K)$ is continuous. The topology on $C^\infty(K)$ is determined by the family of seminorms $\{\rho_{K,n}\}_{n\geq 0}$ defined in equation (E.6). Theorem E.30 therefore implies that there exist $C_K > 0$ and $N_K \geq 0$ such that equation (E.7) holds.

(c) ⇒ (a). Suppose that statement (c) holds, and that $f_k \to 0$ in $C_c^\infty(\mathbb{R})$. Then there exists a compact $K \subseteq \mathbb{R}$ such that $\operatorname{supp}(f_k) \subseteq K$ for all k. Let C_K, N_K be as given in statement (c). Then by equation (E.7), we have

$$|\langle f_k, \mu\rangle| \;\le\; C_K \sum_{n=0}^{N_K} \|f_k^{(n)}\|_\infty \;\to\; 0 \quad \text{as } k \to \infty.$$

Hence μ is continuous. □

Thus, a linear functional μ on $C_c^\infty(\mathbb{R})$ is continuous if each restriction of μ to $C^\infty(K)$ is continuous. Since the topology on any particular $C^\infty(K)$ is given by a countable family of seminorms, for each individual compact K we have the "continuity equals boundedness" characterization given in equation (E.7). However, each compact K can give us possibly different constants C_K and N_K. If a single integer N can be used for all compact K (with possibly different C_K), then the smallest such N is called the *order* of μ. For example, $\delta^{(j)}$ defined as in equation (E.5) belongs to $\mathcal{D}'(\mathbb{R})$ and has order n (see Exercise 4.16).

Additional Problems

E.4. Show that $C_c^\infty(\mathbb{R})$ is not complete with respect to the metric induced from the family of seminorms $\{\rho_{mn}\}_{m,n\ge 0}$, where $\rho_{mn}(f) = \|f^{(n)}\,\chi_{[-m,m]}\|_\infty$.

E.6 The Weak and Weak* Topologies on a Normed Linear Space

The weak topology on a normed space and the weak* topology on the dual of a normed space were introduced in Examples E.7 and E.8. We will study these topologies more closely in this section. They are specific examples of generic "weak topologies" determined by the requirement that a given class of mappings $f_\alpha : X \to Y_\alpha$ be continuous. The "weak topology" corresponding to such a class is the weakest (smallest) topology such that each map f_α is continuous. Another example of such a weak topology is the product topology, which is defined in Section E.7.

E.6.1 The Weak Topology

Let X be a normed space. The topology induced from the norm on X is called the *strong* or *norm topology* on X.

For each $\mu \in X^*$, the functional $\rho_\mu(x) = |\langle x, \mu\rangle|$ for $x \in X$ is a seminorm on X. The topology induced by the family of seminorms $\{\rho_\mu\}_{\mu \in X^*}$ is the weak topology on X, denoted

$$\sigma(X, X^*).$$

By Theorem E.16, X is a locally convex topological vector space with respect to the weak topology. If $\rho_\mu(x) = 0$ for every $\mu \in X^*$, then, by the Hahn–Banach Theorem,

$$\|x\| \;=\; \sup_{\|\mu\|=1} |\langle x, \mu\rangle| \;=\; \sup_{\|\mu\|=1} \rho_\mu(x) \;=\; 0.$$

Hence $x = 0$, so the weak topology is Hausdorff (see Exercise E.17). If we let

$$B_r^\mu(x) \;=\; \bigl\{y \in X : \rho_\mu(x-y) < r\bigr\} \;=\; \bigl\{y \in X : |\langle x-y, \mu\rangle| < r\bigr\}$$

denote the open strips determined by these seminorms, then a base for the topology $\sigma(X, X^*)$ is

$$\mathcal{B} \;=\; \biggl\{\bigcap_{j=1}^{n} B_r^{\mu_j}(x) : n \in \mathbb{N},\; \mu_j \in X^*,\; r > 0,\; x \in X\biggr\}.$$

Convergence with respect to this topology is called weak convergence in X, denoted $x_i \xrightarrow{\text{w}} x$.

Suppose that $\mu \in X^*$ and $\{x_i\}_{i\in I}$ is a net in X such that $x_i \xrightarrow{\text{w}} 0$. By Remark E.20, ρ_μ is continuous with respect to the weak topology, so

$$|\langle x_i, \mu\rangle| \;=\; \rho_\mu(x_i) \;\to\; 0.$$

Thus μ is continuous with respect to the weak topology as well. The next exercise shows that the weak topology is the *smallest* topology with respect to which each $\mu \in X^*$ is continuous.

Exercise E.42. Suppose that X is a normed space, and that $\mathcal{T}$ is a topology on X such that each $\mu \in X^*$ is continuous with respect to $\mathcal{T}$. Show that $\sigma(X, X^*) \subseteq \mathcal{T}$.

Even though the weak topology is weaker than the norm topology, there are a number of situations where we have the surprise that a "weak property" implies a "strong property." For example, Problem E.6 shows that every weakly closed subspace of a normed space is strongly closed, and vice versa.

E.6.2 The Weak* Topology

Let X be a normed space. For each $x \in X$, the functional $\rho_x(\mu) = |\langle x, \mu\rangle|$ for $\mu \in X^*$ is a seminorm on X^*. The topology induced by the family of seminorms $\{\rho_x\}_{x\in X}$ is the weak* topology on X^*, denoted

$$\sigma(X^*, X).$$

By Theorem E.16, X^* is a locally convex topological vector space with respect to the weak* topology. Further, if $\rho_x(\mu) = 0$ for every $x \in X$ then, by definition of the operator norm,

$$\|\mu\| \;=\; \sup_{\|x\|=1} |\langle x, \mu\rangle| \;=\; \sup_{\|x\|=1} \rho_x(\mu) \;=\; 0.$$

Hence $\mu = 0$, so the weak* topology is Hausdorff. If we let

$$B_r^x(\mu) \;=\; \bigl\{\nu \in X^* : \rho_x(\mu - \nu) < r\bigr\} \;=\; \bigl\{\nu \in X^* : |\langle x, \mu - \nu\rangle| < r\bigr\}$$

denote the open strips determined by these seminorms, then a base for the topology $\sigma(X^*, X)$ is

$$\mathcal{B} \;=\; \biggl\{\bigcap_{j=1}^{n} B_r^{x_j}(\mu) : n \in \mathbb{N},\; x_j \in X,\; r > 0,\; \mu \in X^*\biggr\}.$$

Convergence with respect to this topology is called weak* convergence in X^*, denoted $\mu_i \xrightarrow{\text{w*}} \mu$.

Each vector $x \in X$ is identified with the functional $\widetilde{x} \in X^{**}$ defined by

$$\langle \mu, \widetilde{x}\,\rangle \;=\; \langle x, \mu\rangle, \qquad \mu \in X^*.$$

Suppose that $x \in X$ and that $\{\mu_i\}_{i \in I}$ is a net in X^* such that $\mu_i \xrightarrow{\text{w*}} 0$. Then

$$|\langle \mu_i, \widetilde{x}\,\rangle| \;=\; |\langle x, \mu_i\rangle| \;=\; \rho_x(\mu_i) \;\to\; 0,$$

so $\widetilde{x}$ is continuous with respect to the weak* topology. The next exercise shows that the weak* topology is the smallest topology with respect to which $\widetilde{x}$ is continuous for each $x \in X$.

Exercise E.43. Suppose that X is a normed space, and that $\mathcal{T}$ is a topology on X^* such that $\widetilde{x}$ is continuous with respect to $\mathcal{T}$ for each $x \in X$. Show that $\sigma(X^*, X) \subseteq \mathcal{T}$.

Additional Problems

E.5. Let X be a normed space, and let $\mathcal{T}$ be the strong topology on X.

(a) Show directly that $\sigma(X, X^*) \subseteq \mathcal{T}$.

(b) Use the fact that each $\mu \in X^*$ is continuous (by definition) in the strong topology to show that $\sigma(X, X^*) \subseteq \mathcal{T}$.

E.6. Let X be a normed space and S a subspace of X. Prove that the following statements are equivalent.

(a) S is strongly closed (i.e., closed with respect to the norm topology).

(b) S is weakly closed (i.e., closed with respect to the weak topology).

E.7. Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for a Hilbert space H.

(a) Show that $e_n \xrightarrow{\text{w}} 0$.

(b) Show that $n^{1/2} e_n$ does not converge weakly to 0.

(c) Show that 0 belongs to the weak closure of $\{n^{1/2} e_n\}_{n \in \mathbb{N}}$, i.e., 0 is an accumulation point of this set with respect to the weak topology on H.

E.7 Alaoglu's Theorem

If X is a normed space, then the closed unit ball in X or X^* is compact with respect to the norm topology if and only if X is finite-dimensional (Problem A.27). Even so, Alaoglu's Theorem states *that the closed unit ball in* X^* *is compact in the weak* topology.* We will prove this theorem in this section.

E.7.1 Product Topologies

For the case of two topological spaces X and Y, the product topology on $X \times Y$ was defined in Section A.6. We review here some facts about the product topology on arbitrary products of topological spaces.

Definition E.44 (Product Topology). Let J be a nonempty index set, and for each $j \in J$ let X_j be a nonempty topological space. Let X be the Cartesian product of the X_j:

$$X = \prod_{j \in J} X_j = \bigl\{\{x_j\}_{j \in J} : x_j \in X_j \text{ for } j \in J\bigr\},$$

where a sequence $F = \{x_j\}_{j \in J}$ denotes the mapping $F\colon J \to \cup_j X_j$ given by $F(j) = x_j$. The Axiom of Choice states that X is nonempty. The *product topology* on X is the topology generated by the collection

$$\mathcal{B} = \left\{\prod_{j \in J} U_j : U_j \text{ open in } X_j, \text{ and } U_j = X_j \text{ except for finitely many } j\right\}.$$

Since $\mathcal{B}$ is closed under finite intersections, it forms a base for the product topology. Thus, the open sets in X are unions of elements of $\mathcal{B}$.

For each j, we define the *canonical projection* of X onto X_j to be the mapping $\pi_j\colon X \to X_j$ defined by

$$\pi_j\bigl(\{x_i\}_{i \in J}\bigr) = x_j.$$

If U_j is an open subset of X_j and we define $U_i = X_i$ for all $i \neq j$, then

$$\pi_j^{-1}(U_j) = \prod_{i \in J} U_i, \tag{E.8}$$

which is open in X. Hence each π_j is a continuous map. Moreover, if $\mathcal{T}$ is any other topology on X such that each canonical projection π_j is continuous then $\mathcal{T}$ must contain all of the sets given by equation (E.8). Since finite intersections of those sets form the base $\mathcal{B}$, we conclude that $\mathcal{T}$ must contain the product topology on X. Thus, *the product topology on* X *is the weakest topology such that each canonical projection* π_j *is continuous.* This is another example of a general kind of weak topology determined by the requirement that a given class of mappings on X be continuous.

Tychonoff's Theorem is a fundamental result on compact sets in the product topology. The proof uses the Axiom of Choice, see [Fol99]. In fact, Kelley proved that Tychonoff's Theorem is *equivalent* to the Axiom of Choice [Kel50].

Theorem E.45 (Tychonoff's Theorem). *For each $j \in J$, let X_j be a topological space. If each X_j is compact, then $X = \prod_{j\in J} X_j$ is compact in the product topology.*

E.7.2 Statement and Proof of Alaoglu's Theorem

Now we can prove Alaoglu's Theorem (also known as the *Banach–Alaoglu Theorem*).

Theorem E.46 (Alaoglu's Theorem). *Let X be a normed linear space, and let*

$$B^* = \{\mu \in X^* : \|\mu\| \le 1\}$$

be the closed unit ball in X^*. Then B^* is compact in X^* with respect to the weak topology on X^*.*

Proof. For each $x \in X$, let

$$D_x = \{z \in \mathbb{C} : |z| \le \|x\|\}$$

be the closed unit ball of radius $\|x\|$ in the complex plane. Each D_x is compact in $\mathbb{C}$, so Tychonoff's Theorem implies that $D = \prod_{x\in X} D_x$ is compact in the product topology.

The elements of D are sequences $\mu = \{\mu_x\}_{x\in X}$ where $\mu_x \in D_x$ for each x. More precisely, μ is a mapping of X into $\bigcup_{x\in X} D_x = \mathbb{C}$ that satisfies $|\mu_x| \le \|x\|$ for all $x \in X$. Thus μ is a functional on X, although it need not be linear. Since μ is a functional, we adopt our standard notation and write $\langle x, \mu\rangle = \mu_x$. Then we have

$$|\langle x, \mu\rangle| \le \|x\|, \qquad x \in X.$$

If μ is linear then we have $\|\mu\| \le 1$, so $\mu \in B^*$. In fact B^* consists exactly of those elements of D that are linear.

Our next goal is to show that B^* is closed with respect to the product topology restricted to D. Suppose that $\{\mu_i\}_{i\in I}$ is a net in B^* and $\mu_i \to \mu \in D$, where the convergence is with respect to the product topology. Since the canonical projections are continuous in the product topology, we have

$$\langle x, \mu_i\rangle = \pi_x(\mu_i) \to \pi_x(\mu) = \langle x, \mu\rangle, \qquad x \in X. \tag{E.9}$$

In particular, given $x, y \in X$ and $a, b \in \mathbb{C}$, we have

$$\langle ax + by, \mu_i\rangle \to \langle ax + by, \mu\rangle.$$

However, each μ_i is linear since it belongs to B^*, so we also have

$$\langle ax + by, \mu_i \rangle \;=\; a \langle x, \mu_i \rangle + b \langle y, \mu_i \rangle \;\to\; a \langle x, \mu \rangle + b \langle y, \mu \rangle.$$

Therefore $\langle ax + by, \mu \rangle = a \langle x, \mu \rangle + b \langle y, \mu \rangle$, so μ is linear and therefore belongs to B^*. Hence B^* is a closed subset of D. Since D is compact in the product topology, we conclude that B^* is also compact in the product topology (see Problem A.22).

Now we will show that the product topology on D restricted to B^* is the same as the weak* topology on X^* restricted to B^*. To do this, let $\mathcal{T}$ denote the product topology on D restricted to B^*, and let σ denote the weak* topology on X^* restricted to B^*. Suppose that $\{\mu_i\}_{i \in I}$ is a net in B^* and $\mu_i \to \mu$ with respect to the product topology on D. Equation (E.9) shows that $\mu \in B^*$ and $\mu_i \xrightarrow{\text{w}^*} \mu$ in this case. Hence every subset of B^* that is closed with respect to σ is closed with respect to $\mathcal{T}$, so we have $\sigma \subseteq \mathcal{T}$ (see Exercise A.48).

Conversely, fix any $x \in X$ and suppose that $\{\mu_i\}_{i \in I}$ is a net in B^* such that $\mu_i \xrightarrow{\text{w}^*} \mu$. Then $\pi_x(\mu_i) = \langle x, \mu_i \rangle \to \langle x, \mu \rangle = \pi_x(\mu)$, so the canonical projection π_x is continuous with respect to the weak* topology restricted to B^*. However, $\mathcal{T}$ is the weakest topology with respect to which each canonical projection is continuous, so $\mathcal{T} \subseteq \sigma$.

Thus, $\mathcal{T} = \sigma$. Since we know that B^* is compact with respect to $\mathcal{T}$, we conclude that it is also compact with respect to σ. That is, B^* is compact with respect to the weak* topology on X^* restricted to B^*, and this implies that it is compact with respect to the weak* topology on X^*. □

As a consequence, if X is reflexive then the closed unit ball in X^* is weakly compact. In particular, the closed unit ball in a Hilbert space is weakly compact. On the other hand, the space c_0 is not reflexive, and its closed unit ball is not weakly compact (Problem E.9).

E.7.3 Implications for Separable Spaces

Alaoglu's Theorem has some important consequences for separable spaces. Although we will restrict our attention here to normed spaces, many of these results hold more generally, see [Rud91].

We will need the following lemma.

Lemma E.47. *Let $\mathcal{T}_1$, $\mathcal{T}_2$ be topologies on a set X such that:*

(a) *X is Hausdorff with respect to $\mathcal{T}_1$,*

(b) *X is compact with respect to $\mathcal{T}_2$, and*

(c) $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Then $\mathcal{T}_1 = \mathcal{T}_2$.

Proof. Suppose that $F \subseteq X$ is $\mathcal{T}_2$-closed. Then F is $\mathcal{T}_2$-compact since X is $\mathcal{T}_2$-compact (see Problem A.22). Suppose that $\{U_\alpha\}_{\alpha \in J}$ is any cover of F by sets that are $\mathcal{T}_1$-open. Then each of these sets is also $\mathcal{T}_2$-open, so there must exist a finite subcollection that covers F. Hence F is $\mathcal{T}_1$-compact, and therefore is $\mathcal{T}_1$-closed since $\mathcal{T}_1$ is Hausdorff (again see Problem A.22). Consequently, $\mathcal{T}_2 \subseteq \mathcal{T}_1$. □

Now we can show that a weak*-compact subset of the dual space of a separable normed space is metrizable.

Theorem E.48. *Let X be a separable normed space, and fix $K \subseteq X^*$. If K is weak*-compact, then the weak* topology of X^* restricted to K is metrizable.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset of X.

Each $x \in X$ determines a seminorm ρ_x on X^* given by $\rho_x(\mu) = |\langle x, \mu\rangle|$ for $\mu \in X^*$. The family of seminorms $\{\rho_x\}_{x\in X}$ induces the weak* topology $\sigma(X^*, X)$ on X^*. The subfamily $\{\rho_{x_n}\}_{n\in\mathbb{N}}$ also induces a topology on X^*, which we will call $\mathcal{T}$. Since this is a smaller family of seminorms, we have $\mathcal{T} \subseteq \sigma(X^*, X)$.

Suppose that $\mu \in X^*$ and $\rho_{x_n}(\mu) = 0$ for every $n \in \mathbb{N}$. Then we have $\langle x_n, \mu\rangle = 0$ for every n. Since $\{x_n\}_{n\in\mathbb{N}}$ is dense in X and μ is continuous, this implies that $\mu = 0$. Consequently, by Exercise E.17, the topology $\mathcal{T}$ is Hausdorff. Thus $\mathcal{T}$ is a Hausdorff topology induced from a countable family of seminorms, so Exercise E.24 tells us that this topology is metrizable. Specifically, $\mathcal{T}$ is induced from the metric

$$\mathrm{d}(\mu,\nu) \;=\; \sum_{n=1}^{\infty} 2^{-n}\, \frac{\rho_{x_n}(\mu-\nu)}{1+\rho_{x_n}(\mu-\nu)}, \qquad \mu,\nu \in X^*.$$

Let $\mathcal{T}|_K$ and $\sigma(X^*, X)|_K$ denote these two topologies restricted to the subset K. Then we have that K is Hausdorff with respect to $\mathcal{T}|_K$, and is compact with respect to $\sigma(X^*, X)|_K$. Lemma E.47 therefore implies that $\mathcal{T}|_K = \sigma(X^*, X)|_K$. The topology $\mathcal{T}_K$ is metrizable, as it is formed by restricting the metric d to K. Hence $\sigma(X^*, X)|_K$ is metrizable as well. □

In the course of the proof of Theorem E.48 we constructed a metrizable topology $\mathcal{T}$ on X^*, and showed that the restrictions of $\mathcal{T}$ and $\sigma(X^*, X)$ to any weak*-compact set are equal. This does not show that $\mathcal{T}$ and $\sigma(X^*, X)$ are equal. In fact, if X is infinite-dimensional, the weak* topology on X is not metrizable, see [Rud91, p. 70].

Now we can prove a stronger form of Alaoglu's Theorem for separable normed spaces. Specifically, we show that if X is normed and separable, then any bounded sequence in X^* has a weak*-convergent subsequence.

Theorem E.49. *If X is a separable normed space then the closed unit ball in X^* is sequentially weak*-compact. That is, if $\{\mu_n\}_{n\in\mathbb{N}}$ is any sequence in X^* with $\|\mu_n\| \le 1$ for $n \in \mathbb{N}$, then there exists a subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ and $\mu \in X^*$ such that $\mu_{n_k} \xrightarrow{\mathrm{w}^*} \mu$.*

Proof. By Alaoglu's Theorem, the closed unit ball B^* in X^* is weak*-compact. Since X is separable, Theorem E.48 implies that the weak* topology on B^* is metrizable. Finally, since B^* is a compact subset of a metric space, Theorem A.69 implies that it is sequentially compact. □

Corollary E.50. *If X is a reflexive normed space then the closed unit ball in X^* is sequentially weakly compact. In particular, the closed unit ball in a separable Hilbert space is sequentially weakly compact.*

However, even if X is separable, the closed unit ball in X need not be metrizable in the weak topology. For example, [Con90, Prop. V.5.2] shows that this is the case for the closed unit ball in ℓ^1.

There are several deep results on weak compactness that we will not elaborate upon. For example, the *Eberlein-Šmulian Theorem* states that a subset of a Banach space is weakly compact if and only if it is weakly sequentially compact, see [Con90, Thm. V.13.1].

Additional Problems

E.8. This problem will use Alaoglu's Theorem to construct an element of $(\ell^\infty)^*$ that does not belong to ℓ^1.

(a) For each $n \in \mathbb{N}$, define $\mu_n \colon \ell^\infty \to \mathbb{C}$ by $\langle x, \mu_n \rangle = \frac{1}{n}(x_1 + \cdots + x_n)$ for $x = (x_1, x_2, \dots) \in \ell^\infty$. Show that $\mu_n \in (\ell^\infty)^*$ and $\|\mu_n\| \le 1$.

(b) Use Alaoglu's Theorem to show that there exists a $\mu \in (\ell^\infty)^*$ that is an accumulation point of $\{\mu_n\}_{n \in \mathbb{N}}$.

(c) Show that $\mu \ne \widetilde{x}$ for any $x \in \ell^1$, where $\widetilde{x}$ is the image of x under the natural embedding of ℓ^1 into $(\ell^1)^{**} = (\ell^\infty)^*$.

E.9. Show that the closed unit ball $\{x \in c_0 : \|x\|_\infty \le 1\}$ in c_0 is not weakly compact.