Hints and Solution Sketches for Exercises and Additional Problems

Exercises from Chapter 1

1.8 Hint: The Lebesgue Dominated Convergence Theorem.

1.15 Hint: (c) First prove the result assuming that \( f \in C_c(\mathbb{R}) \), so \( f \) is uniformly continuous. Then use the fact that \( C_c(\mathbb{R}) \) is dense in \( L^p(\mathbb{R}) \) for \( p \) finite to approximate an arbitrary \( L^p(\mathbb{R}) \) function by a function in \( C_c(\mathbb{R}) \).

1.17 Hint: Show that \( |e^{2\pi i \eta x} - 1| \leq \min\{2, 2\pi |\eta x|\} \) (see the “proof by picture” in Figure 1.8). Hence, for any \( x \) we have \( |e^{2\pi i \eta x} - 1| \to 0 \) as \( \eta \to 0 \). For \( p < \infty \), apply the Lebesgue Dominated Convergence Theorem.

1.27 Hints: (a) Here are three approaches, all variations on the same theme.

First, show that
\[
|f \ast g(x)| \leq \int \left( |f(y)| |g(x - y)|^{1/p} \right) |g(x - y)|^{1/p'} \, dy,
\]
and apply Hölder’s Inequality with exponents \( p \) and \( p' \) to the two factors.

Second, recall that
\[
\|f \ast g\|_p = \sup\{ |\langle f \ast g, h \rangle| : \|h\|_{p'} = 1 \}.
\]
Show that
\[
|\langle f \ast g, h \rangle| \leq \int |f(y)| \|T_y g|, |h\| \, dy,
\]
and apply Hölder’s Inequality to \( \langle T_y g|, |h\rangle \).

Third, write out \( \|f \ast g\|_p \) as an iterated integral, and apply Minkowski’s Integral Inequality (Problem 1.18).

(b) Show that
\[
|f \ast g(x)| \leq \int \left( |f(y)|^{p/r} |g(x - y)|^{q/r} \right) |f(y)|^{p(\frac{1}{p} - \frac{1}{r})} |g(x - y)|^{q(\frac{1}{q} - \frac{1}{r})} \, dy,
\]
and apply Hölder’s Inequality for a product of three functions (see Problem B.13) using exponents \( r, p_1, p_2 \), where

\[
\frac{1}{p_1} = \frac{1}{p} - \frac{1}{r}, \quad \frac{1}{p_2} = \frac{1}{q} - \frac{1}{r}.
\]

1.36 Hints: (a) To follow the method of Theorem 1.33, show uniform continuity directly by using Hölder’s Inequality and the fact that translation is strongly continuous on \( L^p(\mathbb{R}) \).

To follow the method of Exercise 1.35, use the fact that \( 1 < p < \infty \) implies \( 1 < p' < \infty \), and hence \( C_c(\mathbb{R}) \) is dense in both \( L^p(\mathbb{R}) \) and \( L^{p'}(\mathbb{R}) \).

(b) For a counterexample, consider \( g \) identically 1.

1.37 Solution sketch for \( m = 1 \). We have that

\[
\frac{(f * g)(x + h) - (f * g)(x)}{h} = \int f(y) \frac{g(x + h - y) - g(x - y)}{h} \, dy.
\]

The integrand converges pointwise a.e. to \( f(y) g'(x - y) \) as \( h \to 0 \). Use the Mean Value Theorem to show that, as a function of \( y \),

\[
\left| f(y) \frac{g(x + h - y) - g(x - y)}{h} \right| \leq |f(y)| \|g'\|_\infty \in L^1(\mathbb{R}).
\]

Then apply the Lebesgue Dominated Convergence Theorem.

1.45 Hint: Show that if \( \hat{g}(\xi) \neq 0 \) for a.e. \( \xi \) then \( g \notin g * L^1(\mathbb{R}) \) (examples are the Dirichlet function \( d \) or the Gaussian function \( g(x) = e^{-x^2} \)).

1.55 Hint: Use (or prove) the fact that if \( f \in L^1(\mathbb{R}) \) and \( \varepsilon > 0 \), then there exists a \( \delta > 0 \) such that \( \int_E |f| < \varepsilon \) for any measurable \( E \subseteq \mathbb{R}^d \) satisfying \( |E| < \delta \).

1.61 Hint: Apply an argument similar to the one used in Theorem 1.60, using the fact that Theorem 1.56 implies that the Fundamental Theorem of Calculus holds for \( g \) on every interval \([a, b]\).

1.67 Hints: Apply Hölder’s Inequality to

\[
\int |f(x) - f(x - t)||k_\lambda(t)|^{1/p} \cdot |k_\lambda(t)|^{1/p'} \, dt,
\]

or apply Minkowski’s Integral Inequality.

1.68 Hint: Show that

\[
\|f - f * k_\lambda\|_\infty \leq \int \|f - T_\lambda f\|_\infty |k_\lambda(t)| \, dt,
\]

and then split the integral into \( |t| < \delta \) and \( |t| \geq \delta \).
1.69 Hints: For the last part of the problem, by Exercise 1.36 we know that \( f * k_\lambda \) will be uniformly continuous on \( \mathbb{R} \). Show that if \( f * k_\lambda \to f \) uniformly on \( \mathbb{R} \), then \( f \) is uniformly continuous. However, not every element of \( C_0(\mathbb{R}) \) is uniformly continuous, e.g., consider \( f(x) = \sin x^2 \).

1.77 Hints: (a) \( \Rightarrow \) (b). Choose \( f \in J \) and \( g \in L^1(\mathbb{R}) \). If \( f * g \notin J \) then by the Hahn–Banach Theorem there exists \( \varphi \in L^1(\mathbb{R})^* = L^\infty(\mathbb{R}) \) such that \( \langle h, \varphi \rangle = 0 \) for all \( h \in J \) while \( \langle f * g, \varphi \rangle \neq 0 \).

(b) \( \Rightarrow \) (a). Let \( \{k_\lambda\}_{\lambda > 0} \) be an approximate identity, and consider \( (T_\alpha f) * k_\lambda \).

1.78 Hints: Show that \( J \) is translation-invariant and hence is an ideal.

For the opposite inclusion, consider \( (T_\alpha g) * k_\lambda \) where \( \{k_\lambda\}_{\lambda > 0} \) is an approximate identity.

1.79 Hint: Suppose \( \hat{g}(\xi) = 0 \). Given \( \varepsilon > 0 \), let \( h \in L^1(\mathbb{R}) \) satisfy \( \hat{h}(\xi) = 1 \).

Show that \( h \) cannot be well-approximated by elements of span\( \{T_\alpha g\}_{\alpha \in \mathbb{R}} \).

1.81 Hints: By making a change of variables, using the half-angle formula \( \sin^2 x = \frac{(1 - \cos 2x)}{2} \), and integration by parts, show that

\[
\int w = \int \left( \frac{\sin \pi x}{\pi x} \right)^2 dx = \lim_{R \to \infty} \frac{1}{\pi} \int_{-R}^{R} \frac{\sin x}{x} dx.
\]

Note that while \( \frac{\sin x}{x} \) is not an integrable function, the improper Riemann integral \( \int_0^\infty \frac{\sin x}{x} dx \) does exist and equals \( \frac{\pi}{2} \) (see Problem 1.41).

1.94 Hints: (b) Take the Fourier transform of \( f_k \) and make the change of variables \( \eta = 2\pi k \xi \).

(c) Since \( f_k, \hat{f}_k \in L^1(\mathbb{R}) \), the Inversion Theorem applies.

1.95 Hints: (a) Even though \( \frac{\sin x}{x} \) is not integrable, show that

\[
K = \sup_{0 < a < b < \infty} \left| \int_a^b \frac{\sin x}{x} dx \right| < \infty.
\]

Then use the fact that \( f \) is odd to write \( \hat{f}(\xi) = -2i \int_0^\infty f(x) \sin(2\pi \xi x) dx \).

Substitute this into \( \int_1^b \hat{f}(\xi) d\xi \), and use Fubini’s Theorem to justify interchanging the integrals. Show that \( K \|f\|_1 \) is a bound for the desired supremum.

1.102 Hint: (c) Write \( \Phi(0)^2 = \left( \int e^{-\pi x^2} dx \right) \left( \int e^{-\pi y^2} dy \right) \), and switch to polar coordinates.

1.105 Hint: Break into intervals \( |x| \leq 1 \) and \( |x| > 1 \). For the latter, write \( |f^{(n)}(x)| = |x^2 f^{(n)}(x)|/|x^2| \).
Additional Problems from Chapter 1

1.2 Remark: To say that a function \( f \in L^1(\mathbb{R}) \) is even means that there is an even function \( g \) such that \( f = g \) a.e.

1.4 Hint: Fix \( \xi \) and let \( \alpha = e^{-2\pi i \theta} \) be the complex number of modulus 1 such that \( |\hat{f}(\xi)| = \alpha \hat{f}(\xi) \). Then consider \( \hat{f}(0) - |\hat{f}(\xi)| \).

1.5 Remark. For \( z \notin \mathbb{Z} \), the Gamma function satisfies the functional equation \( \Gamma(z) \Gamma(z-1) \sin \pi z = \pi \). However, \( \sin \pi z \neq 0 \) when \( z \notin \mathbb{Z} \), so \( \Gamma(z) \neq 0 \) for \( z \notin \mathbb{Z} \). Also, for \( z = n \in \mathbb{N} \) we have \( \Gamma(n) = (n-1)! \neq 0 \). Hence \( \Gamma(z) \neq 0 \) for all \( z \) for which it is defined.

1.6 Hint: (c) Apply the Lebesgue Dominated Convergence Theorem to the partial sums of the series.

1.9 Hints: \( \int e^{-2y^2} dy = (\pi/2)^{1/2} \) and \( \int y^2 e^{-2y^2} dy = (\pi/2)^{1/2}/4 \).

1.12 Hint: Set \( f(p) = \ln A_p^2 \) and show that
\[
\begin{align*}
  f(p) &= \frac{(p-1) \ln (p-1) - (p-2) \ln p}{p}, \\
  f'(p) &= \frac{2 - 2\ln p + \ln (p-1)}{p^2}.
\end{align*}
\]
Conclude from this that \( f \) has critical points at
\[
\frac{e^2 \pm e\sqrt{e^2-4}}{2} \approx 1.19243, 6.19662.
\]

1.13 Hint: Consider
\[
\begin{align*}
  f(x) &= \begin{cases} 
    \frac{1}{|x|}, & |x| > 1, \\
    1, & |x| \leq 1
  \end{cases}, \\
  g(x) &= \begin{cases} 
    \frac{1}{\ln |x|}, & |x| > e, \\
    1, & |x| \leq e
  \end{cases}.
\end{align*}
\]
Then \( f \in L^p(\mathbb{R}) \) for \( p > 1 \) and \( g \in C_0(\mathbb{R}) \), but \( (f * g)(x) = \infty \) for every \( x \).

1.20 Hints: (a) Show that \( \chi_E * \chi_{-E} \in C_0(\mathbb{R}) \).

(b) \( E \) cannot have zero measure because \( \cup_{r \in \mathbb{Q}} (E + r) = \mathbb{R} \).

(c) Suppose that \( |A_r| > 0 \), and let \( E \) be as in part (b). Let \( A_r = A \cap (E+r) \) for \( r \in \mathbb{Q} \). Then the \( A_r \) are disjoint sets whose union is \( A \). Use part (b) to show that if \( A_r \) is measurable then it must have measure zero.

1.21 Hint: Set \( F(y) = \int |f(x,y)| \, dx \), and note that \( \|F\|_p \) is the left-hand side of equation (1.18). Write
\[
\|F\|_p^p = \int F(y)^{p-1} F(y) \, dy,
\]
and apply Hölder’s Inequality with exponents \( p' \) and \( p \) to the two factors appearing in the integrand above.
1.24 Hints: (a) Consider a single subinterval \( \{[x, y]\} \) in the definition of absolutely continuity.

(b) To find an absolutely continuous function that is not Lipschitz, consider Exercise 1.55. To find a function of bounded variation that is not absolutely continuous, consider the Cantor–Lebesgue function.

(c) Let \( E = \{(x, y) \in [a, b]^2 : x \leq y\} \). By Fubini’s Theorem, the two iterated integrals

\[
\iint_E f'(x) g'(y) \, dx \, dy = \int_a^b \left( \int_a^y f'(x) \, dx \right) g'(y) \, dy
\]

and

\[
\iint_E f'(x) g'(y) \, dy \, dx = \int_a^b f'(x) \left( \int_x^b g'(y) \, dy \right) \, dx
\]

are equal.

1.26 Hint: (a) Write \( \hat{f}(\xi) = 2 \int_0^{1/2} \cos(6\pi x) \cos(2\pi \xi x) \, dx \), and apply a trigonometric identity to rewrite the integrand as a sum of two cosines. Alternatively, write \( f = \frac{1}{2\pi} \left( M_3 \chi_{[-1/2, 1/2]} + M_{-3} \chi_{[-1/2, 1/2]} \right) \), use the duality between modulation and translation together with Exercise 1.7 to compute \( \hat{f} \), and apply trig identities.

1.27 Let \( g(x) = e^{-x} \chi_{[0, \infty)}(x) \), and observe that \( f'(x) = g(x) - f(x) \).

1.29 Hint: To show \( P \) is unbounded, consider \( f_n = \chi_{[n, n+1]} \). To show \( M \) is unbounded, consider \( f_n(x) = n^{1/p} f(nx) \) for an appropriate \( f \).

1.35 Hint: Consider \( k_\lambda \ast g \).

1.36 Hint: If \( \int k = 0 \), let \( m \in L^1(\mathbb{R}) \) be any function such that \( \int m = 1 \), and consider that both \( \{(k + m)_\lambda \}_{\lambda > 0} \) and \( \{m_\lambda\}_{\lambda > 0} \) are approximate identities.

1.37 Hint: \( g \ast k_\lambda \) belongs to \( g \ast L^1(\mathbb{R}) \).

1.44 Hint: Note that the Inversion Formula is not applicable. Instead, consider the Fourier transform of \( g(x) = (f(x) + f(-x))/2 \) and apply the Uniqueness Theorem.

1.46 Hint: Show that the Inversion Formula applies to \( f(x) = e^{-2\pi |x|} \) (consider Problem 1.1), and that it also applies to \( \chi_{[-1/2, 1/2]} \ast \hat{f} \). Use this to relate the integral in question to \( (\chi_{[-1/2, 1/2]} \ast \hat{f})(1/2) \).

1.48 Hint: Suppose that \( \sum_{k=1}^N T_n g = 0 \) a.e. Take the Fourier transform of both sides, and consider the fact that a nontrivial trigonometric polynomial can have only countably many zeros (see Section F.4).

1.49 Hint: Use the Inversion Formula to write \( f(x + h) - f(x) \) as an integral involving \( \hat{f} \), and then estimate the integral by breaking it into the regions where \( |\xi| \leq 1/|h| \) and \( |\xi| > 1/|h| \).
1.52 Hints: (b) Either compute directly, or show that \((T_1 B_n - B_n)\) = \(\hat{B}_n\) and apply the Uniqueness Theorem. Set \(\chi = \chi_{[0,1]}\) and note that \(\hat{\chi}(\xi) = M_{-1/2} d_x(\xi) = e^{-\pi i \xi \sin \frac{\pi}{2}}\).

(c) Use the Inversion Formula to show that the decay of \(\hat{B}_n\) in frequency implies that \(B_n\) must be smooth in the time variable. To show that \(B_n^{(n-1)}\) is piecewise linear, use the relation proved in part (b).

(d) Note that \(\chi(x) = \chi(2x) + \chi(2x - 1)\). Let \(c_0 = c_1 = 1\) and set \(c_k = 0\) for all other \(k\). Show that \((\chi * \chi)(x) = \frac{1}{2} \sum_{k \in \mathbb{Z}} (c * c)_k \chi(2x - k)\), where \(c * c\) is the discrete convolution of the sequence \(c = (c_k)_{k \in \mathbb{Z}}\) with itself (see Definition 1.48). Note that \(c * c\) has only three nonzero terms.

1.54 Hint: Write \(\text{range}(T) = \bigcup_k T(\mathcal{B}_k(0))\), where \(\mathcal{B}_k(0)\) is the open ball in \(X\) of radius \(k\) centered at the origin. If \(\text{range}(T)\) is nonmeager, then some set \(T(\mathcal{B}_k(0))\) must contain an open ball. Apply Lemma C.102 to conclude that \(\text{range}(T)\) contains an open ball.

1.56 Hint: The technique of Exercise 1.102 carries over to complex parameters.

1.58 Hint: Let \(\{\phi_{\lambda}\}_{\lambda > 0}\) be the Gauss kernel. Show that \((f * \phi_{\lambda})(0) = (f * \phi_{\lambda})^{\vee}(0) = (\hat{f} \phi_{\lambda})^{\vee}(0)\), and apply Fatou’s Lemma:

\[
\int_{|\xi| > R} \liminf_{\lambda \to \infty} \hat{f}(\xi) \phi_{\lambda}(\xi) d\xi \\
\leq \liminf_{\lambda \to \infty} \int \hat{f}(\xi) \phi_{\lambda}(\xi) d\xi - \int_{|\xi| \leq R} \hat{f}(\xi) \phi_{\lambda}(\xi) d\xi
\]

1.59 Hint: \(\hat{g}(\xi) = (4\pi \xi)/(1 + 4\pi^2 \xi^2)\).

1.61 Hint: Apply the product rule \((fg)^{(n)} = \sum_{j=0}^{n} \binom{n}{j} f^{(j)} g^{(n-j)}\).

1.62 Hint: Write \(x^m = ((x - y) + y)^m\) and apply the Binomial Theorem.

1.63 Hint: Let \(K \in C_c^\infty(\mathbb{R})\) be such that \(K(0) = 1\), and construct an approximate identity from \(k = K^\vee\).

1.64 Hint: Construct \(f\) so that for each \(k \in \mathbb{N}\) we have \(f(x) = c_k e^{ikx}\) in a small neighborhood of \(k\).