MATH 1501 Prof. Heil
Fall 2001

LIMITS

Imprecise discussion of limits. The intuitive idea of what we mean when we say $\lim_{x \to c} f(x) = L$ is that as x approaches closer and closer to c, the function values $f(x)$ approach closer and closer to L. But this is not a *precise* statement, it does not tell you how to figure out whether a function has a limit or what that limit is. It's like saying "a hot object in a cold room will cool down"—from that statement you get no precise information about how fast the object is cooling, or when it will reach a particular temperature, etc. By contrast, Newton's law of cooling is a *precise* statement about how the rate of change of the temperature relates to the difference between the temperature of the object and the temperature of the surrounding medium, and this *precise* statement does allow you to calculate how long it will take for an object to cool down, etc. We need a similarly precise definition of the meaning of limit in order to be able to deal *quantitatively* with limits.

Precise definition of limit. We must state EXACTLY what it means to write

$$\lim_{x \to c} f(x) \;=\; L.$$

The EXACT, PRECISE meaning of these symbols is that:

$$\boxed{\begin{array}{c}\text{Given any number } \varepsilon > 0, \text{ there is a number } \delta > 0 \text{ such that:} \\ \text{if } 0 < |x - c| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.\end{array}}$$

Thus, to *prove* that a limit is a certain value L, we must demonstrate that given *any* number $\varepsilon > 0$ it is possible to find a number $\delta > 0$ so that a particular something happens. It's not enough to prove that something for a single value of ε; we must be able to demonstrate that *no matter what* ε *is specified, there does exist a corresponding* δ *so that: if* $0 < |x - c| < \delta$ *then* $|f(x) - L| < \varepsilon$.

Example. Let me prove that $\lim_{x \to 3} (5x - 7) = 8$. I'm going to give the proof only—I'm not going to explain *how* I got the proof, I just want you to see if you can follow the proof and understand each step in the proof that I make. What we must show is that:

$$\begin{array}{c}\text{Given any number } \varepsilon > 0, \text{ there is a number } \delta > 0 \text{ such that:} \\ \text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |(5x - 7) - 8| < \varepsilon.\end{array}$$

We must *prove* this fact, i.e., we must make a logical argument that proceeds step by step to demonstrate that the above statement is true. So, I have to show that given any number $\varepsilon > 0$, there is a number δ so that some particular thing happens. I have to convince you that no matter what number $\varepsilon > 0$ that you consider, there is a corresponding number $\delta > 0$ so that: if $0 < |x-3| < \delta$ then $|(5x-7)-8| < \varepsilon$. Here is my proof, see if you believe that it proves what I claim that it does. If you don't believe that I've done it, you need to email me your questions about it right away.

Proof that $\lim_{x\to 3}(5x-7) = 8$.

Let $\varepsilon > 0$ be any number. I have to convince you that there is a number $\delta > 0$ so that:

$$\text{if } 0 < |x-3| < \delta \quad \text{then} \quad |(5x-7)-8| < \varepsilon.$$

I hope to convince you that the number $\delta = \varepsilon/5$ has this property. How I came up with this value of δ is what scratchwork is for, but the only issue as far as the proof is concerned is whether I can convince you that $\delta = \varepsilon/5$ does indeed have the right property.

So, let me try to show you that the number $\delta = \varepsilon/5$ has the right property. What we have to show is that if x is any number such that $0 < |x-3| < \delta$, then $f(x) = 5x-7$ satisfies $|(5x-7)-8| < \varepsilon$. Now, I can't just *claim* that $f(x)$ has the property, I have to *demonstrate* to you that it does have this property, by using the fact that x has the property $0 < |x-3| < \delta$. Here is the calculation: *If x satisfies $0 < |x-3| < \delta$, then:*

$$\begin{aligned}
|(5x-7)-8)| &= |5x-15| && \text{(arithmetic)} \\
&= 5\,|x-3| && \text{(arithmetic)} \\
&< 5\,\delta && \text{(because } |x-3| < \delta) \\
&= 5\,\frac{\varepsilon}{5} && \text{(because } \delta = \varepsilon/5) \\
&= \varepsilon && \text{(arithmetic).}
\end{aligned}$$

Thus, I've shown that *if $0 < |x-3| < \delta$, then* $|(5x-7)-8| < \varepsilon$. That's exactly what I needed to show, so the proof is done.

Proof that $\lim_{x\to 3}(5x-7) = 8$ (short version).

My proof above contained a lot of extra words that I included just to try to explain what was going on. Here is a shorter version that contains only the essential argument and nothing else:

Let $\varepsilon > 0$ be any number. Let $\delta = \varepsilon/5$. If x satisfies $0 < |x - 3| < \delta$, then:

$$|(5x - 7) - 8)| \;=\; |5x - 15| \;=\; 5\,|x - 3| \;<\; 5\,\delta \;=\; 5\,\frac{\varepsilon}{5} \;=\; \varepsilon.$$

Done.

Sample incorrect proof. Here is an actual proof written by a student, and my comments on what is wrong with it.

Student line 1: Let $\varepsilon > 0$.
Comment. Good start.

Student line 2: If $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$.
Comment. Problem 1: What is δ? This makes no sense unless δ has been given a value. Problem 2: This line is what we have to PROVE. The student wrote it down as if it was a fact. If we knew that this line was true then we would be done! The whole point is to PROVE that if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$! If you like, you could write "WE MUST SHOW THAT if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$", but you cannot simply write it down as if it was a fact.

Student line 3: Choose $\delta = \varepsilon/5$.
Comment. Yes, but you should have told me what δ was BEFORE you tried to use it in line 2.

Student line 4: If $|x - 3| < \delta = \varepsilon/5$ then $|5x - 15| < \varepsilon$.
Comment. Again, this is what must be PROVED. If we knew that this line was true, we would be done. You can't write this down as a fact, you have to PROVE that it is a true fact. You have to *convince me* that if x satisfies $0 < |x - 3| < \delta$ then $f(x) = 5x - 7$ satisfies $|(5x - 7) - 8| < \varepsilon$. This is what the student tries to do in the next line, but he shouldn't have claimed as he did on this line that the conclusion $|5x - 15| < \varepsilon$ was true until he has *proved it.*

Student line 5: $\Rightarrow$ and $|5x - 15| < 5\,|x - 3| = \delta \Rightarrow |5x - 15| < 5\varepsilon/5 = \varepsilon$.
Comment. Problem 1: What does "$\Rightarrow$ and" mean? "implies and"? The wording does not make sense. What you write must make sense—if you said exactly what you wrote out loud, would another person understand what you meant? Problem 2: Where did $5\,|x - 3| = \delta$ come from? We have assumed that x satisfies $|x - 3| < \delta$, how could it follow from this that $5\,|x - 3| = \delta$? (It doesn't.)

A more difficult proof. Let me prove something more difficult now. I'll try to prove that $\lim_{x\to 1} x^3 = 1$. Again, let me just give the proof without the scratchwork. The point here is: Can you follow the proof? Do you believe that it really shows what it is supposed to? Here it is, with extra explanations at each step.

Let $\varepsilon > 0$ be any given value. I have to convince you that there is a number $\delta > 0$ so that:

$$\text{if } 0 < |x - 1| < \delta \quad \text{then} \quad |x^3 - 1| < \varepsilon.$$

I claim that the number $\delta = \min(\varepsilon/7, 1)$ has this property. So, what I have to show you is that for this value of δ, if x has the property $0 < |x - 1| < \delta$, then x^3 has the property that $|x^3 - 1| < \varepsilon$.

So, let's look at those x's which have the property that $0 < |x - 1| < \delta$, and let's try to see whether x^3 will have the right property or not. Look at our x. We know that $0 < |x - 1| < \delta$. We also know that δ is the smaller of the two numbers $\varepsilon/7$ and 1. So, δ could at most be 1, and maybe is smaller. But in any case, we have $|x - 1| < 1$ (and maybe smaller than that). Therefore, x must be somewhere in the range $0 < x < 2$ (and maybe even in a smaller range than that). Hence $0 < x^2 < 4$, and so

$$|x^2 + x + 1| \;\le\; |x^2| + |x| + |1| \;<\; 4 + 2 + 1 \;=\; 7.$$

Thus, for all of these x's, we know that $|x^2 + x + 1| < 7$. Therefore, for these x, the ones that satisfy $0 < |x - 1| < \delta$, we have

$$\begin{aligned}
|x^3 - 1| &= |x - 1|\,|x^2 + x + 1| && \text{(factoring)} \\
&< 7\,|x - 1| && \text{(because } |x^2 + x + 1| < 7 \text{ for those } x\text{'s)} \\
&< 7\,\delta && \text{(because } |x - 1| < \delta) \\
&\le 7\,\frac{\varepsilon}{7} && \text{(because } \delta \le \varepsilon/7) \\
&= \varepsilon && \text{(arithmetic).}
\end{aligned}$$

Thus, I've shown that if $0 < |x - 1| < \delta$, then $|x^3 - 1| < \varepsilon$. That's exactly what I needed to show, so the proof is done.

Scratchwork. HOW did I come up with this proof? That's what scratchwork is for. But the scratchwork isn't part of the proof, it's just what I do in order to figure out how to get the proof. After I've done the scratchwork, I can throw it away, it has no more use. Only the PROOF is important. Most of you are able to do a fairly decent job on the scratchwork; rather, it seems to be the *proof* that is giving trouble. If you are having trouble doing the scratchwork, get help now. But also be absolutely sure that you understand *how* to write the proof, and—more importantly—*WHY* the proof is written the way it is. If you don't understand this, *get help now.*