Perturbations of Banach Frames and Atomic Decompositions

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Abstract. Banach frames and atomic decompositions are sequences that have basis-like properties but which need not be bases. In particular, they allow elements of a Banach space to be written as linear combinations of the frame or atomic decomposition elements in a stable manner. In this paper we prove several functional-analytic properties of these decompositions, and show how these properties apply to Gabor and wavelet systems. We first prove that frames and atomic decompositions are stable under small perturbations. This is inspired by corresponding classical perturbation results for bases, including the Paley–Wiener basis stability criteria and the perturbation theorem of Kato. We introduce new and weaker conditions which ensure the desired stability. We then prove duality properties of atomic decompositions and consider some consequences for Hilbert frames. Finally, we demonstrate how our results apply in the practical case of Gabor systems in weighted L^2 spaces. Such systems can form atomic decompositions for $L_w^2(\mathbb{R})$, but cannot form Hilbert frames for $L_w^2(\mathbb{R})$ unless the weight is trivial.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [DS] as part of their seminal research in nonharmonic Fourier series. Daubechies, Grossmann, and Meyer [DGM] later found a fundamental new application, to wavelet and Gabor transforms. Frames continue to play an important role in each of these areas.

A set of vectors $\{y_i\}$ in a Hilbert space H is a (Hilbert) frame if the norms $\|x\|_H$ and $\|\{\langle x, y_i\rangle\}\|_{\ell^2}$ are equivalent. Define $Ux = \{\langle x, y_i\rangle\}$. Then $U^*Ux = \sum \langle x, y_i\rangle y_i$ is a continuous invertible mapping of H onto itself. With $\tilde{y}_i = (U^*U)^{-1}y_i$, we have the reconstruction formulas

$$(1.1) \qquad x = \sum \langle x, \tilde{y}_i\rangle y_i = \sum \langle x, y_i\rangle \tilde{y}_i.$$

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The collection $\{\tilde{y}_i\}$ also forms a frame, the dual frame of $\{y_i\}$. The representations in equation (1.1) need not be unique, i.e., $\{y_i\}$ need not be a basis. A frame which is a basis must be a Riesz basis. Conversely, all Riesz bases are frames. The basic theory of frames in Hilbert spaces can be found in DUFFIN and SCHAEFFER's original paper [DS], YOUNG's classic text [Y], DAUBECHIES' paper [D1] and book [D2], or the research-tutorial [HW].

Frames were extended to Banach spaces by GRÖCHENIG [G]. In Hilbert spaces, the norm equivalence hypothesis leads immediately to the reconstruction formula (1.1). This does not hold in Banach spaces in general. A decomposition of a Banach space is therefore defined as follows.

Definition 1.1. Let X be a Banach space and let X_d be an associated Banach space of scalar-valued sequences indexed by $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let $\{y_i\}_{i \in \mathbb{N}} \subset X'$ and $\{x_i\}_{i \in \mathbb{N}} \subset X$ be given. If:

(a) $\{\langle x, y_i \rangle\} \in X_d$ for each $x \in X$,

(b) the norms $\|x\|_X$ and $\|\{\langle x, y_i \rangle\}\|_{X_d}$ are equivalent, and

(c) $x = \sum_{i=1}^{\infty} \langle x, y_i \rangle x_i$ for each $x \in X$,

then $(\{y_i\}, \{x_i\})$ is a (linear) *atomic decomposition* of X with respect to X_d. If the norm equivalence is given by $A \|x\|_X \leq \|\{\langle x, y_i \rangle\}\|_{X_d} \leq B \|x\|_X$, then A, B are a choice of *atomic bounds* for $(\{y_i\}, \{x_i\})$.

Atomic decompositions have played a key role in the recent development of wavelet theory and Gabor, or windowed Fourier, analysis. Prior to the discovery of smooth wavelet orthonormal bases for the Hilbert space $L^2(\mathbb{R})$ in [M] and prior to the construction of wavelet and Gabor frames for $L^2(\mathbb{R})$ in [DGM], FRAZIER and JAWERTH had constructed wavelet atomic decompositions for Besov spaces in [FJ], which they termed the "ϕ-transform." FEICHTINGER constructed Gabor atomic decompositions for the modulation spaces in [F2]. These are Banach spaces similar in many respects to Besov spaces, defined by smoothness and decay conditions. WALNUT studied this construction in detail in [W], extending and improving many results for the case of weighted L^2 spaces. In particular, he showed that Gabor atomic decompositions in $L_w^2(\mathbb{R})$ need not be Hilbert frames. FEICHTINGER and GRÖCHENIG have developed a general theory which applies to an extremely broad class of function spaces and group representations [FG1], [FG2], [G].

In this paper we show that several new results on atomic decompositions follow simply once they are considered from a functional analytic point of view. We begin in Section 2 by demonstrating that atomic decompositions and the related concept of Banach frames are stable under small perturbations. This is inspired by corresponding classical perturbation results for bases, e.g., the PALEY–WIENER basis stability criteria [PW], [Y] and the perturbation theorem of KATO [K]. The paper by RETHERFORD and HOLUB [RH] is an excellent survey of perturbation results for bases. We introduce new and weaker conditions which ensure the desired stability. Next, we prove duality properties of atomic decompositions in Section 3, and consider some consequences for Hilbert frames in Section 4. Related Hilbert space results, partially inspired by a weak version of Theorem 2.3 first proved in [He], have appeared in [C1], [C2], [C3], and these have formed the basis for additional results by FAVIER and ZALIK [FZ]. We point out an implication for wavelet frames in $L^2(\mathbb{R})$ by connecting to a result of CHUI and SHI

from [CS]. We also discuss the relationship between the hypotheses in our theorems to unconditional convergence of series, and prove a result on unconditional convergence which HOLUB has used in his recent characterization of near-Riesz bases in Hilbert spaces [Ho]. Finally, in Section 5 we demonstrate how our results can be applied to the practical case of Gabor atomic decompositions in weighted L^2 spaces. We show that no Gabor system can be a Hilbert frame for $L_w^2(\mathbb{R})$ unless the weight is trivial (bounded both above and below). Therefore atomic decompositions are a required tool for studying Gabor systems in weighted L^2 spaces.

The coherent state atomic decompositions are the most important for applications. These include both the wavelet and Gabor cases. A coherent state atomic decomposition arises from a group G acting on a Banach space X via a group representation π. We assume that $\pi(g)$ is a continuous invertible mapping of X onto itself for each $g \in G$. In the wavelet case, the group is the affine group $\mathbf{A} = (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$, and the group representation σ is obtained by forming time-scale translates of functions ψ:

$$\sigma(a,b)\psi(t) \; = \; C(a,b)\,\psi(at-b), \qquad (a,b) \in \mathbf{A},$$

with $C(a,b)$ chosen appropriately for X. For the Gabor case, the group is the Heisenberg group $\mathbf{H} = \mathbf{T} \times \mathbb{R} \times \mathbb{R}$, and the group representation ρ is obtained via time-frequency translates of functions f:

$$\rho(z,a,b)f(t) \; = \; C(z,a,b)\,e^{2\pi ibt}\,f(t-a), \qquad (z,a,b) \in \mathbf{H}.$$

Given such a function space X, group G, and representation π, we select a generator $x \in X$ (often called, by an abuse of terminology, the mother wavelet) and a discrete set $\{g_i\} \subset G$. If $\{\pi(g_i)x\}$ forms an atomic decomposition for X, then it is called a coherent state atomic decomposition. The discrete set $\{g_i\}$ is commonly chosen to be a "regular" subset of the group. For example, a wavelet coherent state decomposition usually has the form $\{a^{n/2}\,\psi(a^n t - mb)\}_{m,n \in \mathbb{Z}}$ while a Gabor coherent state decomposition usually has the form $\{e^{2\pi imbt}\,f(t-na)\}_{m,n \in \mathbb{Z}}$. The relationship of the affine and Heisenberg groups to wavelet and Gabor analysis on $L^2(\mathbb{R})$ is surveyed in detail in [HW].

We will use the following notations and terminology throughout. We assume that sequences are indexed by $\mathbb{N} = \{1, 2, 3, \ldots\}$. We say that a sequence $\{c_i\}$ is finite if only finitely many components are nonzero. A Banach space X_d of sequences is solid if whenever $\{b_i\}$ and $\{c_i\}$ are sequences with $\{c_i\} \in X_d$ and $|b_i| \leq |c_i|$ then it follows that $\{b_i\} \in X_d$ and $\|\{b_i\}\|_{X_d} \leq \|\{c_i\}\|_{X_d}$. For example, let e_j denote the delta sequence $e_j(i) = \delta_{ij}$. If $\{e_j\}_{j \in \mathbb{N}}$ forms an unconditional basis for X_d then X_d is solid. It also follows in this case that X_d has an absolutely continuous norm, i.e., if $\{c_i\} \in X_d$ then $\lim_{n \to \infty} \|\{c_i - c_i \cdot \chi_{I_n}(i)\}\|_{X_d} = 0$, where $\{I_n\}$ is any family of subsets of $\mathbb{N}$ such that $I_1 \subset I_2 \subset \cdots \nearrow \mathbb{N}$ and χ_{I_n} is the characteristic function $\chi_{I_n}(i) = 1$ if $i \in I_n$, 0 if $i \notin I_n$. In particular, the hypotheses on X_d, X_d' in most of our results are satisfied if X_d' can be realized as a Banach space of sequences of scalars and if $\{e_j\}$ forms an unconditional basis for both X_d and X_d'.

2. Perturbation results

An atomic decomposition provides a factorization of the identity map I on X. That is, I is written as a composition of the coefficient mapping $x \mapsto \{\langle x, y_i \rangle\}$ and the reconstruction operator $\{c_i\} \mapsto \sum c_i \, x_i$. Such series reconstructions are theoretically appealing. However, for numerical implementations it is often preferable to formulate the reconstruction operator via an iteration or other algorithm. We therefore make the following definition, which allows freedom in the form of the reconstruction operator.

Definition 2.1. Let X be a Banach space and let X_d be an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{y_i\}_{i \in \mathbb{N}} \subset X'$ and $S \colon X_d \to X$ be given. If:

(a) $\{\langle x, y_i \rangle\} \in X_d$ for each $x \in X$,

(b) The norms $\|x\|_X$ and $\|\{\langle x, y_i \rangle\}\|_{X_d}$ are equivalent, and

(c) S is bounded and linear, and $S\{\langle x, y_i \rangle\} = x$ for each $x \in X$,

then $(\{y_i\}, S)$ is a *Banach frame* for X with respect to X_d. The mapping S is the *reconstruction operator*. If the norm equivalence is given by

$$A \, \|x\|_X \;\leq\; \|\{\langle x, y_i \rangle\}\|_{X_d} \;\leq\; B \, \|x\|_X,$$

then A, B are a choice of *frame bounds* for $(\{y_i\}, S)$.

Note that if $U \colon X \to X_d$ is the coefficient mapping defined by $Ux = \{\langle x, y_i \rangle\}$ then $\|S\|^{-1}$, $\|U\|$ are a choice of frame bounds for the Banach frame $(\{y_i\}, S)$.

We now show that Banach frames are stable under small perturbations of the frame elements.

Theorem 2.2. *Let $(\{y_i\}, S)$ be a Banach frame for X with respect to X_d. Let $\{z_i\} \subset X'$. If there exist λ, $\mu \geq 0$ such that*

(a) $\lambda \, \|U\| + \mu \leq \|S\|^{-1}$, *and*

(b) $\|\{\langle x, y_i - z_i \rangle\}\|_{X_d} \leq \lambda \, \|\{\langle x, y_i \rangle\}\|_{X_d} + \mu \, \|x\|_X$ *for all $x \in X$,*

then there exists a reconstruction operator T such that $(\{z_i\}, T)$ is a Banach frame for X with respect to X_d with frame bounds $\|S\|^{-1} - (\lambda \, \|U\| + \mu)$, $\|U\| + (\lambda \, \|U\| + \mu)$, where U is the coefficient mapping $Ux = \{\langle x, y_i \rangle\}$.

P r o o f. The hypotheses imply that the operator $V \colon X \to X_d$ defined by $Vx = \{\langle x, z_i \rangle\}$ is bounded and satisfies $\|Ux - Vx\|_{X_d} \leq \lambda \, \|Ux\|_{X_d} + \mu \, \|x\|_X$ for all $x \in X$. Therefore,

$$\|Vx\|_{X_d} \;\leq\; (\|U\| \,+\, \lambda \, \|U\| \,+\, \mu) \, \|x\|_X.$$

This establishes the upper frame bound. For the lower bound, observe that $SU = I$, so

$$\|I - SV\| \;\leq\; \|S\| \, \|U - V\| \;\leq\; \|S\| \, (\lambda \, \|U\| \,+\, \mu) \;<\; 1.$$

Therefore SV is invertible, and $\|(SV)^{-1}\| \leq (1 - (\lambda \, \|U\| + \mu) \, \|S\|)^{-1}$. Finally, if we set $T = (SV)^{-1} S$ then $TV = I$, and

$$\|x\|_X \;\leq\; \|T\| \, \|Vx\|_{X_d} \;\leq\; \frac{\|S\|}{1 - (\lambda \, \|U\| + \mu) \, \|S\|} \, \|Vx\|_{X_d}.$$

This gives the desired lower bound: $\left(\|S\|^{-1} - (\lambda \|U\| + \mu) \right) \|x\|_X \leq \|Vx\|_{X_d}.$ □

The hypotheses in Theorem 2.2 are natural from the point of view of perturbation of operators: they mean that the operator $U - V$ is relatively bounded with respect to U (see p. 181 of [K]).

For atomic decompositions, we can perturb in X instead of X'. Our result is a "PALEY–WIENER Theorem for atomic decompositions" (see p. 38 of [Y]).

Theorem 2.3. *Suppose that X_d has an absolutely continuous norm. Let $(\{y_i\}, \{x_i\})$ be an atomic decomposition of X with respect to X_d with bounds A, B. Let $\{w_i\} \subset X$. If there exist λ, $\mu \geq 0$ such that*

(a) $\lambda + \mu B < 1$, *and*

(b) $\left\| \sum c_i (x_i - w_i) \right\|_X \leq \lambda \left\| \sum c_i x_i \right\|_X + \mu \|\{c_i\}\|_{X_d}$ *for any finite sequence $\{c_i\} \in X_d$, then there exists a family $\{z_i\} \subset X'$ such that $(\{z_i\}, \{w_i\})$ is an atomic decomposition of X with respect to X_d with bounds $A (1 + (\lambda + \mu B))^{-1}$, $B (1 - (\lambda + \mu B))^{-1}$. Moreover, $\{w_i\}$ is a basis for X if and only if $\{x_i\}$ is a basis for X.*

Proof. Because of the assumption that X_d has an absolutely continuous norm, the series $\sum \langle x, y_i \rangle w_i$ is convergent for any $x \in X$. If we define $T : X \to X$ by $Tx = \sum \langle x, y_i \rangle w_i$, then

$$\|x - Tx\|_X \leq \lambda \|x\|_X + \mu \|\{\langle x, y_i \rangle\}\|_{X_d} \leq (\lambda + \mu B) \|x\|_X$$

for all $x \in X$. Therefore $\|I - T\| < 1$, so T is invertible. Define $z_i = (T^{-1})^* y_i$. Then

$$x = TT^{-1}x = \sum \langle T^{-1}x, y_i \rangle w_i = \sum \langle x, z_i \rangle w_i.$$

Further,

$$\frac{A}{\|T\|} \|x\|_X \leq A \|T^{-1}x\|_X \leq \|\{\langle T^{-1}x, y_i \rangle\}\|_{X_d} \leq B \|T^{-1}x\|_X \leq B \|T^{-1}\| \|x\|_X,$$

so $(\{z_i\}, \{w_i\})$ is an atomic decomposition of X with respect to X_d. Since $\|T\| \leq 1 + \lambda + \mu B$ and $\|T^{-1}\| \leq (1 - (\lambda + \mu B))^{-1}$, the bounds are as claimed.

Finally, assume that $\{x_i\}$ is a basis for X. Then $\{x_i\}$ and $\{y_i\}$ are biorthonormal, so $Tx_j = \sum \langle T^{-1}Tx_j, y_i \rangle w_i = w_j$. Therefore $\{w_i\}$ is a basis since T is invertible. Conversely, if $\{w_i\}$ is a basis then T^{-1} maps it onto $\{x_i\}$.   □

In the terminology of KATO (p. 181 of [K]), the hypotheses in Theorem 2.3 are that the operator $K : D(K) \subset X_d \to X$ defined by $K\{c_i\} = \sum c_i (x_i - w_i)$ is relatively bounded with respect to the operator $\{c_i\} \mapsto \sum c_i x_i$. It is natural to call K the "perturbation operator," since, as we have seen, conditions on K imply that "$\{w_i\}$ inherits decomposition properties from $\{x_i\}$." We apply Theorem 2.3 to Gabor atomic decompositions for weighted L^2-spaces in Section 5.

Specific choices of λ and μ in Theorem 2.3 give conditions in the style of classic results on basis perturbation.

Corollary 2.4. *Let* $(\{y_i\}, \{x_i\})$ *be an atomic decomposition of* X *with respect to* X_d *with bounds* A, B. *Assume that* X_d, X'_d *satisfy:*

(a) X_d *has an absolutely continuous norm,*

(b) X'_d *is a solid Banach space of scalar-valued sequences, and*

(c) *the action of* $\{c_i\} \in X'_d$ *on* $\{b_i\} \in X_d$ *is given by* $\langle \{b_i\}, \{c_i\} \rangle = \sum b_i \bar{c}_i$.

If $\{w_i\} \subset X$ *is such that* $R = \|\{\|x_i - w_i\|_X\}\|_{X'_d} < B^{-1}$, *then there exists a family* $\{z_i\} \subset X'$ *such that* $(\{z_i\}, \{w_i\})$ *is an atomic decomposition of* X *with respect to* X_d *with bounds* $A(1 + RB)^{-1}$, $B(1 - RB)^{-1}$.

Proof. The hypotheses imply that $\|\sum c_i (x_i - w_i)\|_X \leq R \|\{c_i\}\|_{X_d}$ for any finite sequence $\{c_i\} \in X_d$. Therefore we can apply Theorem 2.3 with $\lambda = 0$ and $\mu = R$. $\square$

A major drawback of Corollary 2.4 is that it generally does not apply to the problem of perturbing the generator of a coherent state atomic decomposition $\{\pi(g_i)x\}$. For example, if $\pi(g)$ is an isometry for each g (as in the standard wavelet or Gabor cases) and if the generator x is perturbed to w, then $\|\pi(g_i)x - \pi(g_i)w\|_X \equiv \|x - w\|_X$ for all i. Hence $\|\{\|\pi(g_i)x - \pi(g_i)w\|_X\}\|_{X'_d}$ will typically be infinite. On the other hand, the hypotheses of Theorem 2.3 may still be applicable. We discuss this further in Sections 4 and 5. Similar remarks apply to the problem of perturbing the discrete set $\{g_i\}$.

We say that a sequence $\{x_i\} \subset X$ is a Bessel sequence for X' with respect to X'_d if there exists a constant D such that

$$\|\{\langle x_i, y \rangle\}\|_{X'_d} \leq D \|y\|_{X'} \quad \text{for all } y \in X'.$$

The constant D is the Bessel bound. The following additional consequence of Theorem 2.3 is motivated by a useful result about Riesz bases in Hilbert spaces [Hi].

Corollary 2.5. *Let* $(\{y_i\}, \{x_i\})$ *be an atomic decomposition of* X *with respect to* X_d *with bounds* A, B, *and such that* $\{x_i\}$ *is a Bessel sequence for* X' *with respect to* X'_d *with Bessel bound* D. *Assume that* X_d, X'_d *satisfy hypotheses* (a), (b), *and* (c) *of Corollary 2.4. Assume that there exists a family* $\{T_k\}$ *of bounded operators on* X *and scalars* a_{ik} *so that*

$$x_i - w_i = \sum_k a_{ik} T_k x_i \quad \text{for each } i.$$

If

(a) $a_k = \sup_i |a_{ik}| < \infty$ *for each* k, *and*

(b) $\sum a_k \|T_k\| < (BD)^{-1}$,

then there exists a family $\{z_i\} \subset X'$ *such that* $(\{z_i\}, \{w_i\})$ *is an atomic decomposition of* X *with respect to* X_d *with bounds* $A(1 + BD \sum a_k \|T_k\|)^{-1}$ *and* $B(1 - BD \sum a_k \|T_k\|)^{-1}$.

Proof. Given a finite sequence $\{c_i\} \in X_d$, we have

$$\left\|\sum_i c_i (x_i - w_i)\right\|_X = \left\|\sum_i c_i \sum_k a_{ik} T_k x_i\right\|_X \leq \sum_k \|T_k\| \left\|\sum_i c_i a_{ik} x_i\right\|_X.$$

Fix any k. Then

$$\left\| \sum_i c_i\, a_{ik}\, x_i \right\|_X \;=\; \sup_{\|y\|_{X'}=1} \left| \sum_i c_i\, a_{ik}\, \langle x_i, y\rangle \right|$$

$$=\; \sup_{\|y\|_{X'}=1} \left| \langle \{c_i\}, \overline{\{a_{ik}\,\langle x_i, y\rangle\}}\rangle \right|$$

$$\leq\; \|\{c_i\}\|_{X_d} \sup_{\|y\|_{X'}=1} \|\{a_{ik}\,\langle x_i, y\rangle\}\|_{X_d'}$$

$$\leq\; D\, a_k\, \|\{c_i\}\|_{X_d},$$

where we have used the fact that X_d' is solid. Hence

$$\left\| \sum_i c_i\,(x_i - w_i) \right\|_X \;\leq\; D\left(\sum_k a_k\, \|T_k\| \right) \|\{c_i\}\|_{X_d}$$

for every finite sequence $\{c_i\} \in X_d$. We can therefore apply Theorem 2.3 with $\lambda = 0$ and $\mu = D \sum a_k\, \|T_k\|$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3. Duality for atomic decompositions

If $\{x_i\}$ is a basis for X with coefficient functionals $\{y_i\}$ then $\{y_i\}$ is a basis for $\overline{\text{span}}\{y_i\} \subset X'$ with coefficient functions $\{x_i\} \subset X''$. We investigate the analogous question for atomic decompositions in this section.

Theorem 3.1. *Let* $(\{y_i\}, \{x_i\})$ *be an atomic decomposition of* X *with respect to* X_d. *Assume* X_d, X_d' *satisfy:*

(a) X_d *is solid,*

(b) X_d' *is a Banach space of scalar-valued sequences,*

(c) *the action of* $\{c_i\} \in X_d'$ *on* $\{b_i\} \in X_d$ *is given by* $\langle \{b_i\}, \{c_i\}\rangle = \sum b_i\, \bar{c}_i$, *and*

(d) X_d' *has an absolutely continuous norm.*

If $\{x_i\}$ *is a Bessel sequence for* X' *with respect to* X_d' *then* $(\{x_i\}, \{y_i\})$ *is an atomic decomposition of* X' *with respect to* X_d'.

Proof. The hypotheses given imply that $\sum c_i\, y_i$ converges in X' for every $\{c_i\} \in X_d'$. In particular, since $\{x_i\}$ is a Bessel sequence, if $y \in X'$ is fixed then $\{\langle x_i, y\rangle\} \in X_d'$, so $\sum \langle x_i, y\rangle\, y_i$ converges in X'. Moreover, if $x \in X$ then

$$\left\langle x,\; \sum_i \langle x_i, y\rangle\, y_i \right\rangle \;=\; \left\langle \sum_i \langle x, y_i\rangle\, x_i,\; y \right\rangle \;=\; \langle x, y\rangle.$$

Hence $\sum \langle x_i, y\rangle\, y_i = y$ for each $y \in X'$.

It remains only to show that there is a constant C such that $C\,\|y\|_{X'} \leq \|\{\langle x_i, y\rangle\}\|_{X'_d}$ for all $y \in X'$. However, if $y \in X'$ then

$$
\begin{aligned}
\|y\|_{X'} &= \sup_{\|x\|_X=1} |\langle x, y\rangle| \\
&= \sup_{\|x\|_X=1} \left|\sum \langle x, y_i\rangle\,\langle x_i, y\rangle\right| \\
&\leq \sup_{\|x\|_X=1} \|\{\langle x, y_i\rangle\}\|_{X_d}\,\|\{\langle x_i, y\rangle\}\|_{X'_d} \\
&\leq B\,\|\{\langle x_i, y\rangle\}\|_{X'_d}.
\end{aligned}
$$

$\square$

The Bessel sequence hypothesis is clearly necessary. For example, suppose $(\{y_i\}, \{x_i\})$ is an atomic decomposition of X with respect to X_d and that $\{w_j\}$ is not a Bessel sequence for X' with respect to X'_d. Define $z_j = 0$ for each j; then

1. $(\{y_i\} \cup \{z_j\}, \{x_i\} \cup \{w_j\})$ is an atomic decomposition of X with respect to X_d, although

2. $(\{x_i\} \cup \{w_j\}, \{y_i\} \cup \{z_j\})$ is not an atomic decomposition of X' with respect to X'_d.

4.   Frame decompositions in Hilbert spaces

In this section we consider the case $X = H$, a separable Hilbert space, and $X_d = \ell^2$. For Hilbert frames, it is customary to use a definition of frame bounds slightly different from the one we gave for Banach frames in Definition 2.1. In particular, if $\{x_i\}$ is a Hilbert frame then the norm equivalence between $\|x\|_H$ and $\|\{\langle x, x_i\rangle\}\|_{\ell^2}$ is usually written

$$(4.1) \qquad A\,\|x\|_H^2 \ \leq \ \sum_i |\langle x, x_i\rangle|^2 \ \leq \ B\,\|x\|_H^2 \quad \text{for all } x \in H,$$

with these A, B called the frame bounds. For clarity, we will refer to A, B given by equation (4.1) as Hilbert frame bounds; they are the squares of the Banach frame bounds given in Definition 2.1. The following result is an immediate consequence of Theorem 2.2, but a direct proof is so simple that we include it.

Proposition 4.1. *Let $\{x_i\}$ be a Hilbert frame with Hilbert frame bounds A, B. Let $\{w_i\} \subset H$. If there is an $R < A$ such that*

$$(4.2) \qquad\qquad \sum_i |\langle x, x_i - w_i\rangle|^2 \ \leq \ R\,\|x\|_H^2 \quad \text{for all } x \in H,$$

then $\{w_i\}$ is a Hilbert frame with Hilbert frame bounds $A\,(1 - \sqrt{R/A})^2$ and $B\,(1 + \sqrt{R/B})^2$.

Proof. The triangle inequality in ℓ^2 states that $\|\{\langle x, x_i \rangle\}\|_{\ell^2} \le \|\{\langle x, x_i - \omega_i \rangle\}\|_{\ell^2} + \|\{\langle x, \omega_i \rangle\}\|_{\ell^2}$. Therefore, $\sum_i |\langle x, \omega_i \rangle|^2 \ge (A^{1/2} - R^{1/2})^2 \|x\|_H^2$, and a similar proof applies for the upper bound. $\square$

Despite its simplicity, Proposition 4.1 is quite useful for applications. In most cases it is more difficult to verify the lower frame condition than the upper one. Proposition 4.1 shows that the difficult problem reduces to the easier one in the case of perturbation: the family $\{w_i\}$ is a frame if the difference $\{x_i - w_i\}$ satisfies the upper condition with a sufficiently small bound. This is a weaker hypothesis than the standard basis-type assumption that $\sum \|x_i - w_i\|_H^2 < A$. In particular, this latter hypothesis cannot be applied to the problem of perturbing the generator x of a coherent state frame $\{\pi(g_i)x\}$ in typical cases, e.g., if if $\pi(g)$ is an isometry for all g. However, Proposition 4.1 does apply: it states that $\{\pi(g_i)w\}$ is a frame if the set of coherent states $\{\pi(g_i)(x - w)\}$ generated by $x - w$ is a Bessel sequence with bound less than A. FAVIER and ZALIK [FZ] apply Proposition 4.1 explicitly to the case of Gabor and wavelet frames for $L^2(\mathbb{R})$. There are other applications of Proposition 4.1 to problems in irregular sampling and wavelet theory in [FZ] and [C3].

Proposition 4.1 connects with additional known results about frames. For example, using Remark 1 of [CS] we obtain the following: if $\{a^{n/2} f(a^n x - mb)\}_{m,n \in \mathbb{Z}}$ is a wavelet frame for $L^2(\mathbb{R})$ with bounds A, B, and if $g \in L^2(\mathbb{R})$ is a function such that $\mathrm{supp}(\hat{f} - \hat{g}) \subseteq [-\pi, \pi]$, and there exists a number $R < A$ such that

$$\forall \gamma, \quad \frac{1}{b} \sum_n |\hat{f}(a^n \gamma) - \hat{g}(a^n \gamma)|^2 \ \le \ R,$$

then $\{a^{n/2} g(a^n x - mb)\}_{m,n \in \mathbb{Z}}$ is also a frame for $L^2(\mathbb{R})$.

We have already remarked on the importance of the perturbation operator K. For Hilbert frames we are able to prove another result where K plays the main role.

Theorem 4.2. Let $\{x_i\}$ be a Hilbert frame for H, and let $\{w_i\} \subset H$. If $K\{c_i\} = \sum c_i (w_i - x_i)$ is compact as an operator from ℓ^2 into H, then $\{w_i\}$ is a Hilbert frame for $\overline{\mathrm{span}}\{w_i\}$.

Proof. Define $T: \ell^2 \to H$ by $T\{c_i\} = \sum c_i x_i$. Since $\{x_i\}$ is a frame, we know that T is bounded. In fact, $\|T\|^2 \le B$, the upper Hilbert frame bound for $\{x_i\}$. Hence $V = T + K$ is a bounded operator from ℓ^2 into H. If $x \in H$ then we compute

$$\sum |\langle x, w_i \rangle|^2 \ = \ \|V^* x\|_H^2 \ \le \ \|T + K\|^2 \|x\|_H^2 \ \le \ B \left(1 + \frac{\|K\|}{\sqrt{B}}\right) \|x\|_H^2.$$

This establishes that $\{w_i\}$ satisfies an upper frame bound. The hypothesis that K is compact will give the existence of the lower frame bound, but not a concrete value for it.

By Theorem 2.1 of [C1], to show the existence of the lower frame bound for $\{w_i\}$ it suffices to show that the "frame operator" VV^* for $\{w_i\}$ is surjective. Now,

$$VV^* \ = \ S + TK^* + KT^* + KK^*,$$

where $S = TT^*$ is the frame operator for $\{x_i\}$. The operator

$$(TK^* + KT^* + KK^*) \, S^{-1}$$

is compact, so the operator $(TK^* + KT^* + KK^*) \, S^{-1} + I$ has closed range (see Theorem 4.23 of [R]). Composing this with S, we see that VV^* also has closed range.

Now consider VV^* as an operator on the closed subspace $\overline{\operatorname{span}}\{w_i\}$. Here VV^* is injective: if $x \in \overline{\operatorname{span}}\{w_i\}$ and $VV^*x = 0$ then $\sum |\langle x, w_i \rangle|^2 = \langle VV^*x, x \rangle = 0$, whence $x = 0$. Since VV^* has a closed range we therefore have $\operatorname{Range}(VV^*) = (N(VV^*))^\perp = \overline{\operatorname{span}}\{w_i\}$. Thus VV^* is surjective, as desired, and hence $\{w_i\}$ is a frame for $\overline{\operatorname{span}}\{w_i\}$. $\qquad\square$

In particular, $\{w_i\}$ is a frame for $\overline{\operatorname{span}}\{w_i\}$ if $\sum \|x_i - w_i\|_H^2 < \infty$. By Proposition 4.1, we know that if $\sum \|x_i - w_i\|_H^2 < A$ (the lower Hilbert frame bound for $\{x_i\}$) then $\{w_i\}$ is a frame for H, and therefore $\overline{\operatorname{span}}\{w_i\} = H$. However, if we have merely the equality $\sum \|x_i - w_i\|_H^2 = A$, it may happen that $\overline{\operatorname{span}}\{w_i\} \neq H$. For example, let $\{x_i\}$ be an orthonormal basis for H, and set $w_1 = 0$, $w_i = x_i$ for $i > 1$.

Also, note that the condition (4.2) in Proposition 4.1 is precisely the statement that $\|K\| < \sqrt{A}$. If $\|K\| \geq \sqrt{A}$ then $\{w_i\}$ need not be a frame for $\overline{\operatorname{span}}\{w_i\}$. For example, if $\{x_i\}$ is an orthonormal basis for H and we set $w_i = x_i + x_{i+1}$, then $\|K\| = A = 1$ but $\{w_i\}$ is not a frame for $\overline{\operatorname{span}}\{w_i\} = H$.

Our next result establishes the relation between Hilbert frames and atomic decompositions in Hilbert spaces. Note that if $\{y_i\}$ is a Hilbert frame for H then $(\{y_i\}, \{\tilde{y}_i\})$ is an atomic decomposition of H with respect to ℓ^2, where $\{\tilde{y}_i\}$ is the dual frame of $\{y_i\}$. The converse requires additional hypotheses.

Theorem 4.3. *Let $(\{y_i\}, \{x_i\})$ be an atomic decomposition of H with respect to ℓ^2. Then the following statements hold.*

(a) $\{y_i\}$ is a Hilbert frame for H.

(b) If $\{x_i\}$ is a Bessel sequence for H with respect to ℓ^2 then it is a Hilbert frame for H.

(c) Assume $\{x_i\}$ is a Bessel sequence for H with respect to ℓ^2. Define $U, V \colon H \to \ell^2$ by $Ux = \{\langle x, x_i \rangle\}$ and $Vx = \{\langle x, y_i \rangle\}$. Then $\{x_i\}$ is the dual frame of $\{y_i\}$ if and only if $\operatorname{Range}(U) = \operatorname{Range}(V)$.

Proof. Statement (a) follows immediately from the definition. For (b), the lower frame bound follows from Theorem 3.1, or directly from the computation

$$\|x\|_H^4 \;=\; \left(\sum_i \langle x, y_i \rangle \, \langle x_i, x \rangle\right)^2 \leq \sum_i |\langle x, y_i \rangle|^2 \sum_i |\langle x_i, x \rangle|^2 \;\leq\; B^2 \, \|x\|_H^2 \sum_i |\langle x_i, x \rangle|^2 .$$

Finally, for (c), note that the reconstruction formula (1.1) implies $U^*V = V^*U = I$. Let $E = \operatorname{Range}(U)$. Since U is injective and $UV^*U = U$, we have $(UV^*)|_E = I|_E$. If $E = \operatorname{Range}(V)$, this implies $UV^*V = V$. Therefore, given $x \in H$,

$$\{\langle x, y_i \rangle\} \;=\; Vx \;=\; UV^*Vx \;=\; \{\langle V^*Vx, x_i \rangle\} \;=\; \{\langle x, V^*Vx_i \rangle\} .$$

In particular, we must have $y_i = V^* V x_i$, whence $x_i = (V^* V)^{-1} y_i$ and $\{x_i\}$ is the dual frame of $\{y_i\}$. Conversely, if $\{x_i\}$ is the dual frame of $\{y_i\}$ then $V x = U V^* V x$, so $\text{Range}(V) = \text{Range}(U)$ since $V^* V$ is invertible. $\square$

It need not be the case that $\{x_i\}$ is the dual frame of $\{y_i\}$ even if $(\{y_i\}, \{x_i\})$ is an atomic decomposition of H with respect to ℓ^2 and $\{x_i\}$ is a Bessel sequence. For example, let $\{y_i\}$ and $\{z_j\}$ be two frames for H. Define $w_j = 0$ for each j. Then $(\{y_i\} \cup \{z_j\}, \{\tilde{y}_i\} \cup \{w_j\})$ is an atomic decomposition of H with respect to ℓ^2, but the dual frame of $\{y_i\} \cup \{z_j\}$ is $\{\tilde{y}_i\} \cup \{\tilde{z}_j\}$.

We close this section with a note about convergence, which generalizes Proposition 6.2.19 of [He]. The hypotheses on X_d, X_d' used in most of our results were needed to ensure that series such as $\sum c_i y_i$ converge unconditionally for every $\{c_i\}$ in the appropriate sequence space. In the Hilbert setting, we know that if $\{y_i\}$ is a Hilbert frame then $\sum c_i y_i$ converges unconditionally in H for every $\{c_i\} \in \ell^2$. In fact, this is true if $\{y_i\}$ is merely a Bessel sequence for H with respect to ℓ^2.

Moreover, if $\{y_i\}$ is an arbitrary sequence in H and $\sum c_i y_i$ converges unconditionally, then Orlicz' Theorem (see p. 18 of [LT]) implies that $\sum |c_i|^2 \|y_i\|_H^2 = \sum \|c_i y_i\|_H^2 < \infty$. Therefore, if $\{y_i\}$ is norm-bounded below (meaning $\inf \|y_i\|_H > 0$), then $\sum |c_i|^2 < \infty$.

In particular, if $\{y_i\}$ is a Bessel sequence for H with respect to ℓ^2 and $\{y_i\}$ is norm-bounded below, then

$$(4.3) \qquad \{c_i\} \in \ell^2 \quad \Longleftrightarrow \quad \sum_i c_i y_i \text{ converges unconditionally in } H.$$

Among other results, Holub [Ho] recently used our equation (4.3) to prove the following equivalence for a Hilbert frame $\{y_i\}$ which is norm-bounded below: $\{y_i\}$ is a near Riesz basis (a Riesz basis plus finitely many elements) if and only if

$$\sum_i c_i y_i \text{ converges in } H \quad \Longleftrightarrow \quad \sum_i c_i y_i \text{ converges unconditionally in } H.$$

It would be useful to obtain similar characterizations in the Banach space setting.

5. Applications to weighted L^2–spaces

A function $w \colon \mathbb{R} \to (0, \infty)$ is called a moderate weight if there exists a submultiplicative function m, i.e., $m(s + t) \le m(s)\, m(t)$ for all s, t, such that $w(s + t) \le m(s)\, w(t)$ for all s, t. Frequently one also assumes that $w(0) = 1$, but this assumption will play no role in the sequel. Corresponding to a moderate weight w is a weighted L^2-space

$$L^2_w(\mathbb{R}) \;=\; \left\{ f : \|f\|_{L^2_w} = \left(\int_{-\infty}^{\infty} |f(t)|^2\, w(t)\, dt \right)^{1/2} < \infty \right\}.$$

$L^2_w(\mathbb{R})$ is a Hilbert space with respect to the weighted inner product

$$\langle f, g \rangle_w \;=\; \int f(t)\, \overline{g(t)}\, w(t)\, dt.$$

It can also be considered as a Banach space with dual space $L^2_{1/w}(\mathbb{R})$ when the duality is defined by $\langle f, g \rangle = \int f(t) \overline{g(t)} \, dt$ for $f \in L^2_w(\mathbb{R})$ and $g \in L^2_{1/w}(\mathbb{R})$. $L^2_w(\mathbb{R})$ is translation invariant, i.e., $f \in L^2_w(\mathbb{R})$ implies $T_a f \in L^2_w(\mathbb{R})$ for any $a \in \mathbb{R}$, where $T_a f(t) = f(t - a)$. The same is true for the modulation operator $E_b f(t) = e^{2\pi i b t} f(t)$. Both E_b and T_a are bounded operators on $L^2_w(\mathbb{R})$.

Given $g \in L^2_w(\mathbb{R})$ and fixed a, b, the Gabor system generated by g is $\{g_{mb,na}\}_{m,n \in \mathbb{Z}}$, where $g_{p,q}(t) = E_p T_q g(t) = e^{2\pi i p t} g(t - q)$. Gabor systems in $L^2(\mathbb{R})$ are discussed in detail in [BHW], [D1], [D2], [HW]. Gabor systems in weighted L^2 spaces and in the more general setting of the modulation spaces are discussed in [BW], [F2], [W].

Gabor systems are very useful in analyzing the local frequency content of a function. FEICHTINGER and WALNUT have shown that Gabor systems can be useful outside the setting of $L^2(\mathbb{R})$. However, as we now show, although $L^2_w(\mathbb{R})$ is a Hilbert space, no Gabor system can form a Hilbert frame for $L^2_w(\mathbb{R})$ unless w is essentially bounded, in which case $L^2_w(\mathbb{R}) = L^2(\mathbb{R})$.

Lemma 5.1. *If the Gabor system $\{g_{mb,na}\}$ is a Hilbert frame for $L^2_w(\mathbb{R})$ then w is bounded above and below.*

Proof. Let A, B be the frame bounds for $\{g_{mb,na}\}$. Let I be any interval of length $1/b$, and let $f \in L^2_w(\mathbb{R})$ be any function supported within I. Then the Plancherel formula for Fourier series implies

(5.1)
$$\sum_{m,n} |\langle f, g_{mb,na} \rangle_w|^2 = \sum_{m,n} \left| \int_I f(t) \overline{g(t - na)} \, w(t) \, e^{-2\pi i m b t} \, dt \right|$$
$$= \sum_n \int_I |f(t) \, g(t - na) \, w(t)|^2 \, dt.$$

Set $G(t) = \sum |g(t - na)|^2$. Considering equation (5.1) together with the frame bounds, we conclude that $A \le G(t) \, w(t) \le B$ a.e. on I, and therefore the same inequalities hold almost everywhere on $\mathbb{R}$. Since G is nonnegative and a-periodic, we can find a subset $E \subset [-a/2, a/2]$ with positive measure such that $0 < \alpha \le G(t) \le \beta < \infty$ for almost every $t \in E$. Also, since w is moderate there must exist a constant $C > 0$ such that

$$\sup_{[na-a/2, na+a/2]} w(t) \le C \inf_{[na-a/2, na+a/2]} w(t) \quad \text{for every } n,$$

with C independent of n [W]. Since G is a-periodic, we therefore have

$$\sup_{[na-a/2, na+a/2]} w(t) \le C \inf_{E+na} w(t) \le \frac{C}{\alpha} \inf_{E+na} G(t) \, w(t) \le \frac{BC}{\alpha}.$$

Hence $w(t) \le BC/\alpha$ a.e., and similarly $w(t) \ge A/(C\beta)$ a.e.   □

Although Hilbert frames for $L^2_w(\mathbb{R})$ are not possible for nontrivial weights, WALNUT [BW], [W], building on ideas of FEICHTINGER [F2], has shown that it is possible to construct Gabor atomic decompositions for $L^2_w(\mathbb{R})$ in much the same spirit as the classical results on Gabor frames for $L^2(\mathbb{R})$. These atomic decompositions will not be

Hilbert frames for $L_w^2(\mathbb{R})$. Yet as atomic decompositions they still provide all of the desirable features of frames, including the equivalence between the norm of a function $f \in L_w^2(\mathbb{R})$ and a sequence space norm of $\{\langle f, g_{m,n} \rangle\}$. The natural sequence space for Gabor atomic decompositions of $X = L_w^2(\mathbb{R})$ is the weighted sequence space $X_d = \ell_v^2$, where the discrete weight v is defined by

$$v(m,n) \;=\; \inf_{[na-a/2, na+a/2]} w(t).$$

We can apply our perturbation results to Gabor atomic decompositions. Let

$$Y \;=\; \left\{ f : \int_{-\infty}^{\infty} |f(t)| \, \|T_t\| \, dt < \infty \right\}.$$

This is a Banach space with respect to the natural norm. Spaces of this type have been defined and studied by FEICHTINGER and GRÖCHENIG [F1], [FG1], [FG2] in great detail; they are Wiener amalgam spaces. In their notation, $Y = W\big(L^1, L_{\|T_{(\cdot)}\|}\big)$.

The following result is Corollary 2.3.8 of [W]:

Proposition 5.2. *Let $g \in Y$ and let $b \le 1$. Then there exists a constant C such that*

$$\left\| \sum_{m,n} a_{m,n} \, g_{mb,na} \right\|_{L_w^2} \;\le\; C \, b^{-1/2} \, \|g\|_Y \, \|\{a_{m,n}\}\|_{\ell_v^2}$$

for all sequences $\{a_{m,n}\} \in \ell_v^2$.

The value of C is independent of g and b and can be estimated explicitly. Applying Theorem 2.3 with $\lambda = 0$ and $\mu = C \, b^{-1/2} \, \|g - h\|_Y$, we obtain the following result, which improves a theorem of WALNUT by giving an explicit bound on $\|g - h\|_Y$.

Corollary 5.3. *Assume that $\{g_{mb,na}\}$ is an atomic decomposition for $L_w^2(\mathbb{R})$ with respect to ℓ_v^2, with bounds A, B. If $g - h \in Y$ and $\|g - h\|_Y < b^{1/2}(BC)^{-1}$, then $\{h_{mb,na}\}$ is also an atomic decomposition of $L_w^2(\mathbb{R})$.*

Acknowledgements

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ERRATA

Note: This errata listing is not included in the published version of this paper.

In the statement of Theorem 2.2 part (a), the inequality $\leq$ should be replaced by $<$.