There are 15 problems on 2 pages of this homework assignment.

Let $R$ be a relation from a set $A$ to a set $B$. The inverse relation from $B$ to $A$, denoted by $R^{-1}$, is the set of ordered pairs $\{(b,a) \mid (a,b) \in R\}$. The complementary relation $\overline{R}$ is the set of ordered pairs $\{(a,b) \mid (a,b) \notin R\}$.

Let $R$ be as above and suppose $S$ is a relation from the set $B$ to a set $C$. The composite of $R$ and $S$, denoted $S \circ R$, is the relation consisting of ordered pairs $(a,c)$ where $a \in A$, $c \in C$, and there exists $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$.

Now let $R$ be a relation on a set $A$. We say that $R$ is irreflexive if for every $a \in A$, $(a,a) \notin R$.

Try finding the inverse relation and complementary relation for all the relations on ICA 5. Which ones can you compose? What is the composite relation? Which ones are irreflexive?

1. Let $R$ be a relation on the set $A$.
   
   (a) Prove or disprove: $R$ could be neither reflexive nor irreflexive.
   
   (b) Prove that $R$ is symmetric if and only if $R = R^{-1}$.
   
   (c) Prove that $R$ is antisymmetric if and only if $R \cap R^{-1}$ is a subset of the diagonal relation $\Delta = \{(a,a) \mid a \in A\}$.
   
   (d) Prove that $R$ is reflexive if and only if $\overline{R}$ is irreflexive.
   
   (e) Prove or disprove: if $R$ is irreflexive, then so is $R^2 = R \circ R$.

2. Suppose that $R$ and $S$ are reflexive relations on a set $A$. Prove or disprove the following.
   
   (a) $R \cup S$ is reflexive.
   
   (b) $R \cap S$ is reflexive.
   
   (c) $R \setminus S$ is reflexive.
   
   (d) $S \circ R$ is reflexive.

3. Let $R$ be the relation on the set of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $ad = bc$.
   
   (a) Prove that $R$ is an equivalence relation.
   
   (b) What is the equivalence class of $(1,2)$ under this relation?
   
   (c) Are there finitely many equivalence classes? Justify your answer.
   
   (d) Give a description of each different equivalence class.

4. Find the smallest equivalence relation on the set $\{a,b,c,d,e\}$ that contains the relation $\{(a,b),(a,c),(d,e)\}$. What does “smallest” mean here?

5. Let $R$ be the relation defined on $\mathbb{Z}$ by $a R b$ if and only if $a^2 \equiv b^2 \pmod{5}$. Prove that $R$ is an equivalence relation and determine the distinct equivalence classes.

6. Prove that propositional equivalence is an equivalence relation on the set of all compound propositions.
7. Suppose that $R_1$ and $R_2$ are equivalence relations on the set $S$. Prove or disprove the following.
   (a) $R_1 \cup R_2$ is an equivalence relation.
   (b) $R_1 \cap R_2$ is an equivalence relation.

8. Let $R_2$ be the relation on $\mathbb{Z}$ defined by $a R_2 b$ if and only if $a \equiv b \pmod{2}$ and $R_3$ be the relation on $\mathbb{Z}$ defined by $a R_3 b$ if and only if $a \equiv b \pmod{3}$. If $R_2 \cup R_3$ is an equivalence relation, find its equivalence classes. Likewise for $R_2 \cap R_3$.

9. Let $a, b, c, d, n,$ and $m$ be integers with $n, m \geq 2$. Prove the following.
   (a) If $a \equiv b \pmod{n}$, then $a^2 \equiv b^2 \pmod{n}$.
   (b) If $a \equiv b \pmod{n}$, then $ac \equiv bc \pmod{n}$.
   (c) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$.
   (d) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.
   (e) If $a \equiv b \pmod{n}$ and $a \equiv c \pmod{n}$, then $b \equiv c \pmod{n}$.
   (f) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a - c \equiv b - d \pmod{n}$.
   (g) If $m \mid n$ and $a \equiv b \pmod{n}$, then $a \equiv b \pmod{m}$.
   (h) If $c > 0$ and $a \equiv b \pmod{n}$, then $ac \equiv bc \pmod{cn}$.

10. Let $a, b, c,$ and $n$ be integers with $n \geq 2$. Show that $ac \equiv bc \pmod{n}$ does not necessarily imply that $a \equiv b \pmod{n}$.

11. Let $a \in \mathbb{Z}$. First, prove that $3 \mid a^2$ if and only if $3 \mid a$. Then prove that $\sqrt{3}$ is irrational.

12. Prove that the product of an irrational number and a nonzero rational number is irrational.

13. Prove that when an irrational number is divided by a (nonzero) rational number, the resulting number is irrational.

14. Let $a$ be an irrational number and $r$ a nonzero rational number. Prove that if $s$ is any real number, then either $ar + s$ or $ar - s$ is irrational.

15. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.