Let \( n_1, n_2, n_3, \ldots \), be a strictly increasing sequence of positive integers. Then the sequence \((t_k)_{k \in \mathbb{N}}\) where \( t_k = s_{n_k} \) is called a subsequence of the sequence \((s_n)\). If \((s_{n_k})\) converges, or diverges to \(+\infty\) or to \(-\infty\), then this limit is called a subsequential limit of \((s_n)\).

We define

\[
\limsup s_n = \lim_{N \to \infty} \sup \{s_n \mid n > N\}
\]

and

\[
\liminf s_n = \lim_{N \to \infty} \inf \{s_n \mid n > N\}.
\]

If \( s_n \) is not bounded above, \( \sup \{s_n \mid n > N\} = +\infty \) for all \( N \) and we define \( \limsup s_n = +\infty \).
Similarly, if \( s_n \) is not below above, \( \inf \{s_n \mid n > N\} = -\infty \) for all \( N \) and we define \( \liminf s_n = -\infty \).

1. Consider the sequences for \( n \in \mathbb{N} \) defined as follows:
\[
a_n = (-1)^n, \quad b_n = \frac{1}{n}, \quad c_n = n^2, \quad d_n = \frac{6n + 4}{7n - 3}.
\]
(a) For each sequence, give an example of a monotone subsequence.
(b) For each sequence, give its set of subsequential limits.
(c) For each sequence, give its \( \limsup \) and \( \liminf \).
(d) Which of the sequences converges? diverges to \(+\infty\)? diverges to \(-\infty\)?
(e) Which of the sequences is bounded?

2. Repeat problem 1 for the sequences:
\[
s_n = \cos(n\pi/3), \quad t_n = \frac{3}{4n + 1}, \quad u_n = (-\frac{1}{2})^n, \quad v_n = (-1)^n + \frac{1}{n}.
\]

3. (a) Let \((s_n)_{n \in \mathbb{N}}\) be a sequence with a subsequence \((s_{n_k})\). Prove by induction that \( n_k \geq k \) for all \( k \in \mathbb{N} \).
(b) Prove that if the sequence \((s_n)\) converges, then every subsequence converges to the same limit.