Sets and Relations

2.1 Sets

Any branch of science, like a foreign language, has its own terminology. Isomorphism, cyclotomic, and coset aren’t words used except in a mathematical context. On the other hand, quite a number of common English words—field, complex, function—have precise mathematical meanings quite different from their usual ones. Students of French or Spanish know that memory work is a fundamental part of their studies; it is perfectly obvious to them that if they don’t know what the words mean their ability to learn grammar and to communicate will be severely hindered. It is, however, not always understood by science students that they too must memorize the terminology of their discipline. Without constant review of the meanings of words, one’s understanding of a paragraph of text or the words of a teacher is very limited. We advise readers of this book to maintain and constantly review a mathematical vocabulary list. The authors have included their own such list in a glossary at the back of this book.

What would it be like to delve into a dictionary if you didn’t already know the meanings of some of the words in it? Most people, at one time or another, have gone to a dictionary in search of a word only to discover that the definition uses another unfamiliar word. Some reflection indicates that a dictionary can be of no use unless there are some words that are so basic that we can understand them without definitions. Mathematics is the same way. There are a few basic terms that we accept without definitions.

Most of mathematics is based on the single undefined concept of set, which we think of as just a collection of things called elements or members. Primitive humans discovered the set of natural numbers with which they learned to count. The set of natural numbers, which is denoted with a capital boldface \( \mathbb{N} \) or, in handwriting, with this symbol, \( \mathbb{N} \), consists of the numbers 1, 2, 3, ..., (the three dots meaning "and so on").\(^1\) The elements of \( \mathbb{N} \) are, of course, just the positive integers. The full set of integers, denoted \( \mathbb{Z} \) or \( \mathbb{Z} \), consists of the natural numbers, their negatives, and 0. We might describe this set by \( \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \). Our convention, which is not universal, is that 0 is an integer, but not a natural number.

\(^1\)Since the manufacture of boldface symbols such as \( \mathbb{N} \) is a luxury not afforded users of chalk or pencil, it has long been traditional to use \( \mathbb{N} \) on blackboards or in handwritten work as the symbol for the natural numbers and to call \( \mathbb{N} \) a blackboard bold symbol.
There are various ways to describe sets. Sometimes it is possible to list the elements of a set within braces.

- \{\text{egg}1, \text{egg}2\} is a set containing two elements, \text{egg}1 and \text{egg}2.
- \{x\} is a set containing one element, \(x\).
- \(N = \{1, 2, 3, \ldots\}\) is the set of natural numbers.
- \(Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}\) is the set of integers.

On other occasions, it is convenient to describe a set with set builder notation. This has the format

\[\{x \mid x \text{ has certain properties}\},\]

which is read "the set of \(x\) such that \(x\) has certain properties." We read "such that" at the vertical line, \(\mid\).

More generally, we see

\[\{\text{some expression} \mid \text{the expression has certain properties}\}.\]

Thus, the set of odd natural numbers could be described as

\[\{n \mid n \text{ is an odd integer, } n > 0\}\]

or as

\[\{2k - 1 \mid k = 1, 2, 3, \ldots\}\]

or as

\[\{2k - 1 \mid k \in N\}.

The expression "\(k \in N\)" is read "\(k\) belongs to \(N\)," the symbol \(\in\) denoting set membership. Thus, "\(m \in Z\)" simply says that \(m\) is an integer. Recall that a slash (/) written over any mathematical symbol negates the meaning of that symbol. So, in the same way that \(\pi \neq 3.14\), we have \(0 \notin N\).

The set of common fractions—numbers like \(\frac{3}{4}, \frac{\sqrt{2}}{\sqrt{3}},\) and \(\frac{5}{7}\)—are ratios of integers with nonzero denominators—is more properly called the set of rational numbers and is denoted \(Q\) or \(\mathbb{Q}\). Formally,

\[Q = \left\{\frac{m}{n} \mid m, n \in Z, n \neq 0\right\}.

The set of all real numbers is denoted \(R\) or \(\mathbb{R}\). To define the real numbers properly requires considerable mathematical maturity. For our purposes, we think of real numbers as numbers that have decimal expansions of the form \(a, a_1a_2\ldots\), where \(a\) is an integer and \(a_1, a_2, \ldots\) are integers between 0 and 9 inclusive. In addition to the rational numbers, whose decimal expansions terminate or repeat, the real numbers include numbers like \(\sqrt{2}, \sqrt{17}, e, \pi, \ln 5\), and \(\cos \frac{\pi}{6}\) whose decimal expansions neither terminate nor repeat. Such numbers are called irrational. An irrational number is a number that cannot be written in the form \(\frac{m}{n}\) with \(m\) and \(n\) both integers. Incidentally, it can be very difficult to decide whether a given real number is irrational. For example, it is unknown whether such numbers as \(e + \pi\) or \(\frac{\pi}{\pi}\) are irrational.

The complex numbers, denoted \(C\) or \(\mathbb{C}\), have the form \(a + bi\), where \(a\) and \(b\) are real numbers and \(i^2 = -1\); that is,

\[C = \{a + bi \mid a, b \in R, i^2 = -1\}.

Sometimes people are surprised to discover that a set can be an element of another set. For example, \([[a, b], c]\) is a set with two elements, one of which is \([a, b]\) and the other \(c\).
Let $S$ denote the set $\{a, b, c\}$. True or false?

(a) $a \notin S$.
(b) $\{a\} \subset S$.

Equality of Sets

Sets $A$ and $B$ are equal, and we write $A = B$, if and only if $A$ and $B$ contain the same elements or neither set contains any element.

**Example 1**

- $\{1, 2, 1\} = \{1, 2\} = \{2, 1\}$;
- $\{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}\} = \{\frac{1}{2}\}$;
- $\{t \mid t = r - s, r, s \in \{0, 1, 2\}\} = \{-2, -1, 0, 1, 2\}$.

The Empty Set

One set that arises in a variety of different guises is the set that contains no elements. Consider, for example, the set $\text{small}$ of people less than 1 millimeter in height, the set $\text{large}$ of people taller than the Eiffel Tower, the set

$\emptyset \subseteq \text{COLLECTOR} = \{n \in \mathbb{N} \mid 5n = 2\}$,

and the set

$S = \{n \in \mathbb{N} \mid n^2 + 1 = 0\}$.

These sets are all equal since none of them contains any elements. The unique set that contains no elements is called the empty set. Set theorists originally used 0 (zero) to denote this set, but now it is customary to use a 0 with a slash through it, $\emptyset$, to avoid confusion between zero and a capital "Oh."

True or false? $\emptyset = \emptyset$.

Subsets

A set $A$ is a subset of a set $B$, and we write $A \subseteq B$, if and only if every element of $A$ is an element of $B$. If $A \subseteq B$ but $A \neq B$, then $A$ is called a proper subset of $B$ and we write $A \subset B$.

When $A \subseteq B$, it is common to say "$A$ is contained in $B" as well as "$A$ is a subset of $B." The notation $A \subset B$, which is common, unfortunately means $A \subset B$ to some people and $A \subseteq B$ to others. For this reason, we avoid it, while reiterating that it is present in a lot of mathematical writing. When you see it, make an effort to discover what the intended meaning is.

We occasionally see "$B \supset A," read "$B$ is a superset of $A." This is an alternative way to express "$A \subseteq B," A is a subset of $B$, just as "$y \geq x" is an alternative way to express "$x \leq y. We generally prefer the subset notation.

**Example 2**

- $\{a, b\} \subseteq \{a, b, c\}$
- $\{a, b\} \not\subseteq \{a, b, c\}$
- $\{a, b\} \subseteq \{a, b, \{a, b\}\}$
- $\{a, b\} \in \{a, b, \{a, b\}\}$
- $\mathbb{N} \not\subseteq \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$
Note the distinction between $A \subseteq B$ and $A \subseteq B$, the latter expressing the negation of $A \subseteq B$; for example,

$\{a, b\} \subseteq \{a, b, c\} \not\subseteq \{a, b, x\}$.

### 2.1.3 Proposition

**Proof**

For any set $A$, $A \subseteq A$ and $\emptyset \subseteq A$.

If $a \in A$, then $a \in A$, so $A \subseteq A$. The proof that $\emptyset \subseteq A$ is a classic model of proof by contradiction. If $\emptyset \subseteq A$ is false, then there must exist some $x \in \emptyset$ such that $x \notin A$. This is an absurdity since there is no $x \in \emptyset$.

### Pause 3

True or false?

(a) $\emptyset \in \{\emptyset\}$

(b) $\emptyset \subseteq \{\emptyset\}$

(c) $\emptyset \subseteq \{\emptyset\}$

(As Shakespeare once wrote, “Much ado about nothing.”)

The following proposition is an immediate consequence of the definitions of “subset” and “equal sets,” and it illustrates the way we prove two sets are equal in practice.

### 2.1.4 Proposition

If $A$ and $B$ are sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Two assertions are being made here.

$(\rightarrow)$ If $A = B$, then $A$ is a subset of $B$ and $B$ is a subset of $A$.

$(\leftarrow)$ If $A$ is a subset of $B$ and $B$ is a subset of $A$, then $A = B$.

Remember that another way to state Proposition 2.1.4 is to say that, for two sets to be equal, it is necessary and sufficient that each be a subset of the other.

Note the distinction between membership, $a \in b$, and subset, $a \subseteq b$. By the former statement, we understand that $a$ is an element of the set $b$; by the latter, that $a$ is a set each of whose elements is also in the set $b$.

### Example 3

Each of the following assertions is true.

- $\{a\} \in \{x, y, \{a\}\}$
- $\{a\} \subseteq \{x, y, a\}$
- $\{a\} \not\subseteq \{x, y, \{a\}\}$
- $\{a, b\} \subseteq \{a, b\}$
- $\emptyset \in \{x, y, \emptyset\}$
- $\emptyset \subseteq \{x, y, \emptyset\}$
- $\emptyset \notin \{x, y, \emptyset\}$

### The Power Set

An important example of a set, all of whose elements are themselves sets, is the power set of a set.

### 2.1.5 Definition

The power set of a set $A$, denoted $\mathcal{P}(A)$, is the set of all subsets of $A$:

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}$$

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2Note the use of lowercase letters for sets, which is not common but certainly permissible.
EXAMPLE 4

- If $A = \{a\}$, then $\mathcal{P}(A) = \{\emptyset, \{a\}\}$.
- If $A = \{a, b\}$, then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.
- $\mathcal{P}\{a, b, c\} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

1. Both statements are true. The set $S$ contains the set $\{a\}$ as one of its elements, but not the element $a$.
2. This statement is false: $\{\emptyset\}$ is not the empty set for it contains one element, the set $\emptyset$.
3. (a) True: $\{\emptyset\}$ is a set that contains the single element $\emptyset$.
   (b) True: The empty set is a subset of any set.
   (c) False: There is just one element in the set $\{\emptyset\}$, (that is, $\emptyset$), and this is not an element of the set $\{\emptyset\}$, whose only element is $\emptyset$.

True/False Questions

(Answers can be found in the back of the book.)

1. $5 \in \{x + 2y \mid x \in \{0, 1, 2\}, y \in \{-2, 0, 2\}\}$
2. $-5 \in \{x + 2y \mid x \in \{0, 1, 2\}, y \in \{-2, 0, 2\}\}$
3. If $A = \{a, b\}$, then $b \subseteq A$.
4. If $A = \{a, b\}$, then $\{a\} \in A$.
5. $\emptyset \subseteq \{\emptyset, \{\emptyset\}\}$
6. $\emptyset \in \{\emptyset, \{\emptyset\}\}$
7. $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$
8. (Assume $A$, $B$, $C$ are sets.) $A \in B$, $B \in C \rightarrow A \subseteq C$.
9. (Assume $A$, $B$ are sets.) $A \subseteq B \rightarrow B \subseteq A$.
10. If $A$ has two elements, then $\mathcal{P}(A)$ has eight elements.

Exercises

The answers to exercises marked [BB] can be found in the Back of the Book.

1. List the (distinct) elements in each of the following sets:
   (a) $\{x \in \mathbb{R} \mid x^2 = 5\}$
   (b) $\{x \in \mathbb{Z} \mid xy = 15 \text{ for some } y \in \mathbb{Z}\}$
   (c) $\{x \in \mathbb{Q} \mid x(x^2 - 2)(2x + 3) = 0\}$
   (d) $\{x + y \mid x \in \{-1, 0, 1\}, y \in \{0, 1, 2\}\}$
   (e) $\{a \in \mathbb{N} \mid a < -4 \text{ and } a > 4\}$

2. List five elements in each of the following sets:
   (a) $\{x + bi \mid a, b \in \mathbb{Z}, i^2 = -1\}$
   (b) $\{a + b\sqrt{2} \mid a \in \mathbb{N}, -b \in \{2, 5, 7\}\}$
   (c) $\left\{\frac{x}{y} \mid x, y \in \mathbb{R}, x^2 + y^2 = 25\right\}$
   (d) $\{n \in \mathbb{N} \mid n^2 + n \text{ is a multiple of } 3\}$

3. Let $A = \{1, 2, 3, 4\}$. List all the subsets $B$ of $A$ such that
   (a) $\{1, 2, 3\} \subseteq B$;
   (b) $\emptyset \subseteq \{1, 2, 3\}$;
   (c) $\emptyset \subseteq B$;
   (d) $\{1, 2\} \subseteq B$;
   (e) $\{1, 2\} \subseteq B$;
   (f) $\{1, 2\} \subseteq B$;
   (g) $\{1, 2\} \subseteq B$.

4. [BB] Let $A = \{a, b\}$. Are the following statements true or false? Explain your answer.
   (a) $a \in A$.
   (b) $A \subseteq A$.
   (c) $\{a, b\} \subseteq A$.
   (d) There are two elements in $A$.

5. Determine which of the following are true and which are false. Justify your answers.
   (a) $\{3, 5\} \subseteq \{1, 3, 5\}$
   (b) $\{1, 3, 5\} \not\subseteq \{3, 5\}$
   (c) $\{1, 3, 5\} \subseteq \{3, 5\}$
   (d) $\{1, 3, 5\} \subseteq \{3, 5\}$
   (e) $\{1, 3, 5\} \subseteq \{3, 5\}$
   (f) $1 \in \{a + b\mid a, b \text{ even integers}\}$
   (g) $0 \in \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}, b \neq 0\}$

6. Find the power sets of each of the following sets:
   (a) $\emptyset$ (b) $\emptyset$ (c) $\emptyset, \{\emptyset\}$

7. Determine whether each of the following statements is true or false. Justify your answers.
   (a) $\emptyset \subseteq \emptyset$ (b) $\emptyset \subseteq \emptyset$
2.2 Operations on Sets

In this section, we discuss ways in which two or more sets can be combined to form a new set.

2.2.1 Definitions

The union of sets $A$ and $B$, written $A \cup B$, is the set of elements in $A$ or in $B$ (or in both). The intersection of $A$ and $B$, written $A \cap B$, is the set of elements that belong to both $A$ and $B$.

**Example 5**

- If $A = \{a, b, c\}$ and $B = \{a, x, y, b\}$, then
  \[
  A \cup B = \{a, b, c, x, y\}, \quad A \cap B = \{a, b\},
  \]
  \[
  A \cup \{\emptyset\} = \{a, b, c, \emptyset\} \quad \text{and} \quad B \cap \{\emptyset\} = \emptyset.
  \]
- For any set $A$, $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

As with addition and multiplication of real numbers, the union and intersection of sets are associative operations. To say that set union is associative is to say that

\[
(A_1 \cup A_2) \cup A_3 = A_1 \cup (A_2 \cup A_3)
\]

for any three sets $A_1, A_2, A_3$. It follows that the expression

\[
A_1 \cup A_2 \cup A_3
\]
is unambiguous. The two different interpretations (corresponding to different insertions of parentheses) agree. The union of \( n \) sets \( A_1, A_2, \ldots, A_n \) is written

\[
A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \quad \text{or} \quad \bigcup_{i=1}^{n} A_i
\]

and represents the set of elements that belong to one or more of the sets \( A_i \). The intersection of \( A_1, A_2, \ldots, A_n \) is written

\[
A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \quad \text{or} \quad \bigcap_{i=1}^{n} A_i
\]

and denotes the set of elements which belong to all of the sets.

Do not assume from the expression \( A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \) that \( n \) is actually greater than 3 since the first part of this expression—\( A_1 \cup A_2 \cup A_3 \)—is present only to make the general pattern clear; a union of sets is being formed. The last term—\( A_n \)—indicates that the last set in the union is \( A_n \). If \( n = 2 \), then \( A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \) means \( A_1 \cup A_2 \). Similarly, if \( n = 1 \), the expression \( A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \) simply means \( A_1 \).

While parentheses are not required in expressions like (1) or (2), they are mandatory when both union and intersection are involved. For example, \( A \cap (B \cup C) \) and \( (A \cap B) \cup C \) are, in general, different sets. This is probably most easily seen by the use of the Venn diagram shown in Fig. 2.1.

The diagram indicates that \( A \) consists of the points in the regions labeled 1, 2, 3, and 4; \( B \) consists of those points in regions 3, 4, 5, and 6 and \( C \) of those in 2, 3, 5, and 7. The set \( B \cup C \) consists of points in the regions labeled 3, 4, 5, 6, 2, and 7. Notice that \( A \cap (B \cup C) \) consists of the points in regions 2, 3, and 4. The region \( A \cap B \) consists of the points in regions 3 and 4; thus, \( (A \cap B) \cup C \) is the set of points in the regions labeled 3, 4, 2, 5, and 7. The diagram enables us to see that, in general, \( A \cap (B \cup C) \neq (A \cap B) \cup C \) and it shows how we could construct a specific counterexample: We could let \( A, B, \) and \( C \) be the sets

\[
A = \{1, 2, 3, 4\}, \quad B = \{3, 4, 5, 6\}, \quad C = \{2, 3, 5, 7\}
\]

as suggested by the diagram and then calculate

\[
A \cap (B \cup C) = \{2, 3, 4\} \neq \{2, 3, 4, 5, 7\} = (A \cap B) \cup C.
\]

There is a way to rewrite \( A \cap (B \cup C) \). In Fig. 2.1, we see that \( A \cap B \) consists of the points in the regions labeled 3 and 4 and that \( A \cap C \) consists of the points in 2 and 3. Thus, the points of \( (A \cap B) \cup (A \cap C) \) are those of 2, 3, and 4. These are just the points of \( A \cap (B \cup C) \) (as observed previously), so the Venn diagram makes it easy to believe that, in general,

\[
A \cap (B \cup C) = (A \cap B) \cup (A \cap C).
\]

While pictures can be helpful in making certain statements seem plausible, they should not be relied on because they can also mislead. For this reason, and because there are situations in which Venn diagrams are difficult or impossible to create, it is important to be able to establish relationships among sets without resorting to a picture.

**Problem 6.** Let \( A, B, \) and \( C \) be sets. Verify equation (3) without the aid of a Venn diagram.
Solution. As observed in Proposition 2.1.4, to show that two sets are equal it is sufficient to show that each is a subset of the other. Here this just amounts to expressing the meaning of \( \cup \) and \( \cap \) in words.

To show \( A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \), let \( x \in A \cap (B \cup C) \). Then \( x \) is in \( A \) and also in \( B \cup C \). Since \( x \in B \cup C \), either \( x \in B \) or \( x \in C \). This suggests cases.

Case 1: \( x \in B \).
In this case, \( x \) is in \( A \) as well as in \( B \), so it's in \( A \cap B \).

Case 2: \( x \in C \).
Here \( x \) is in \( A \) as well as in \( C \), so it's in \( A \cap C \).

We have shown that either \( x \in A \cap B \) or \( x \in A \cap C \). By definition of union, \( x \in (A \cap B) \cup (A \cap C) \), completing this half of our proof.

Conversely, we must show \( (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \). For this, let \( x \in (A \cap B) \cup (A \cap C) \). Then either \( x \in A \cap B \) or \( x \in A \cap C \). Thus, \( x \) is in both \( A \) and \( B \) or in both \( A \) and \( C \). In either case, \( x \in A \). Also, \( x \) is in either \( B \) or \( C \); thus, \( x \in B \cup C \). So \( x \) is both in \( A \) and in \( B \cup C \); that is, \( x \in A \cap (B \cup C) \). This completes the proof. □

PROBLEM 7. For sets \( A \) and \( B \), prove that \( A \cap B = A \) if and only if \( A \subseteq B \).

Solution. Remember that there are two implications to establish and that we use the symbolism \((\rightarrow)\) and \((\leftarrow)\) to mark the start of the proof of each implication.

\((\rightarrow)\) Here we assume \( A \cap B = A \) and must prove \( A \subseteq B \). For this, suppose \( x \in A \). Then, \( x \in A \cap B \) (because we are assuming \( A = A \cap B \)). Therefore, \( x \) is in \( A \) and in \( B \), in particular, \( x \) is in \( B \). This proves \( A \subseteq B \).

\((\leftarrow)\) Now we assume \( A \subseteq B \) and prove \( A \cap B = A \). To prove the equality of \( A \cap B \) and \( A \), we must prove that each set is a subset of the other. By definition of intersection, \( A \cap B \) is a subset of \( A \), so \( A \cap B \subseteq A \). On the other hand, suppose \( x \in A \). Since \( A \subseteq B \), \( x \) is in \( B \) too; thus, \( x \) is in both \( A \) and \( B \). Therefore, \( A \subseteq A \cap B \). Therefore, \( A = A \cap B \). □

For sets \( A \) and \( B \), prove that \( A \cup B = B \) if and only if \( A \subseteq B \).

**Set Difference**

The set difference of sets \( A \) and \( B \), written \( A \setminus B \), is the set of those elements of \( A \) that are not in \( B \). The complement of a set \( A \) is the set \( A^c = U \setminus A \), where \( U \) is some universal set made clear by the context.

**Example 8**

- \( \{a, b, c\} \setminus \{a, b\} = \{c\} \)
- \( \{a, b, c\} \setminus \{a, x\} = \{b, c\} \)
- \( \{a, b, \emptyset\} \setminus \emptyset = \{a, b, \emptyset\} \)
- \( \{a, b, \emptyset\} \setminus \{\emptyset\} = \{a, b\} \)

If \( A \) is the set [Monday, Tuesday, Wednesday, Thursday, Friday], the context suggests that the universal set is the days of the week, so \( A^c = \{\text{Saturday}, \text{Sunday}\} \).

Notice that \( A \setminus B = A \cap B^c \) and also that \( (A^c)^c = A \). For example, if \( A = \{x \in \mathbb{Z} \mid x^2 > 0\} \), then \( A^c = \{0\} \) (it being understood that \( U = \mathbb{Z} \)) and so

\[
(A^c)^c = \{0\}^c = \{x \in \mathbb{Z} \mid x \neq 0\} = A.
\]

You may have previously encountered standard notation to describe various types of intervals of real numbers.
2.2.3 Definition

Interval Notation If \( a \) and \( b \) are real numbers with \( a < b \), then

\[
\begin{align*}
[a, b] &= \{ x \in \mathbb{R} \mid a \leq x \leq b \} & \text{closed} \\
(a, b) &= \{ x \in \mathbb{R} \mid a < x < b \} & \text{open} \\
(a, b] &= \{ x \in \mathbb{R} \mid a < x \leq b \} & \text{half open} \\
[a, b) &= \{ x \in \mathbb{R} \mid a \leq x < b \} & \text{half open}.
\end{align*}
\]

As indicated, a closed interval is one that includes both endpoints, an open interval includes neither, and a half-open interval includes just one endpoint. A square bracket indicates that the adjacent endpoint is in the interval. To describe infinite intervals, we use the symbol \( \infty \) (which is just a symbol) and make obvious adjustments to our notation. For example,

\[
(-\infty, b] = \{ x \in \mathbb{R} \mid x \leq b \},
\]

\[
(a, \infty) = \{ x \in \mathbb{R} \mid x > a \}.
\]

The first interval here is half open; the second is open.

Pause 5

If \( A = [-4, 4] \) and \( B = [0, 5] \), then \( A \setminus B = [-4, 0) \). What is \( B \setminus A \)? What is \( A^c \)?

2.2.4 The Laws of De Morgan

The following two laws, of wide applicability, are attributed to Augustus De Morgan (1806–1871), who, together with George Boole (1815–1864), helped to make England a leading center of logic in the nineteenth century.\(^3\)

\[
\begin{align*}
(A \cup B)^c &= A^c \cap B^c, \\
(A \cap B)^c &= A^c \cup B^c
\end{align*}
\]

Readers should be struck by the obvious connection between these laws and the rules for negating and and or compound sentences described in Section 0.1. We illustrate by showing the equivalence of the first law of De Morgan and the rule for negating "\( A \) or \( B \)."

Problem 9. Prove that \((A \cup B)^c = A^c \cap B^c\) for any sets \( A \), \( B \), and \( C \).

Solution. Let \( A \) be the statement "\( x \in A \)" and \( B \) be the statement "\( x \in B \)." Then

\[
\begin{align*}
x \in (A \cup B)^c & \iff \neg(x \in A \cup B) \\
& \iff \neg(A \text{ or } B) & \text{definition of union} \\
& \iff \neg A \text{ and } \neg B & \text{rule for negating "or"} \\
& \iff x \in A^c \text{ and } x \in B^c \\
& \iff x \in A^c \cap B^c & \text{definition of intersection}.
\end{align*}
\]

The sets \((A \cup B)^c\) and \(A^c \cap B^c\) contain the same elements, so they are the same.

2.2.5 Definition

The symmetric difference of two sets \( A \) and \( B \) is the set \( A \oplus B \) of elements that are in \( A \) or in \( B \), but not in both.

\(^3\)As pointed out by Rudolf and Gerda Fritsch (Der Vierfarbensatz, B. J. Wissenschaftsverlag, Mannheim, 1994 and English translation, The Four-Color Theorem, by J. Peschke, Springer-Verlag, 1998), it was in a letter from De Morgan to Sir William Rowan Hamilton that the question giving birth to the famous Four-Color Theorem was first posed. See Section 3.2 for a detailed account of this theorem, whose proof was found relatively recently after over 100 years of effort!
Readers should note that the symbol $\Delta$, as in $A \Delta B$, is also used to denote symmetric difference.

Notice that the symmetric difference of sets can be expressed in terms of previously defined operations. For example,

$$A \oplus B = (A \cup B) \setminus (A \cap B)$$

and

$$A \oplus B = (A \setminus B) \cup (B \setminus A).$$

**Example 10**

- $\{a, b, c\} \oplus \{x, y, a\} = \{b, c, x, y\}$
- $\{a, b, c\} \oplus \emptyset = \{a, b, c\}$
- $\{a, b, c\} \oplus \emptyset = \{a, b, c, \emptyset\}$

**Problem 11.** Use a Venn diagram to illustrate the plausibility of the fact that $\oplus$ is an associative operation; that is, use a Venn diagram to illustrate that for any three sets $A$, $B$, and $C$,

$$ (A \oplus B) \oplus C = A \oplus (B \oplus C). $$

**Solution.** With reference to Fig. 2.1 again, $A \oplus B$ consists of the points in the regions labeled 1, 2, 5, and 6 while $C$ consists of the points in the regions 2, 3, 5, and 7. Thus, $(A \oplus B) \oplus C$ is the set of points in the regions 1, 3, 6, and 7. On the other hand, $B \oplus C$ consists of the regions 2, 7, 4, and 6 and $A$, of regions 1, 2, 3, 4. Thus, $A \oplus (B \oplus C)$ also consists of the points in regions 1, 3, 6, and 7.

As a consequence of (4), the expression $A \oplus B \oplus C$, which conceivably could be interpreted in two ways, is in fact unambiguous. Notice that $A \oplus B \oplus C$ is the set of points in an odd number of the sets $A$, $B$, $C$: Regions 1, 6, and 7 contain the points of just one of the sets while region 3 consists of points in all three. More generally, the symmetric difference $A_1 \oplus A_2 \oplus A_3 \oplus \cdots \oplus A_n$ of $n$ sets $A_1, A_2, A_3, \ldots, A_n$ is well defined and, as it turns out, is the set of those elements which are members of an odd number of the sets $A_i$. (See Exercise 20 of Section 5.1.)

**The Cartesian Product of Sets**

There is yet another way in which two sets can be combined to obtain another.

### 2.2.6 Definitions

If $A$ and $B$ are sets, the **Cartesian product** (sometimes also called the **direct product**) of $A$ and $B$ is the set

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

(We say "$A$ cross $B$" for "$A \times B$.") More generally, the Cartesian product of $n \geq 2$ sets $A_1, A_2, \ldots, A_n$ is

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

When all the sets are equal to the same set $A$, $A \times A \times \cdots \times A$ is written $A^n$. ♦

The elements of $A \times B$ are called **ordered pairs** because their order is important: $(a, b) \neq (b, a)$ (unless $a = b$). The elements $a$ and $b$ are the **coordinates** of the ordered pair $(a, b)$; the first coordinate is $a$ and the second is $b$. The elements of $A^n$ are called $n$-tuples.
Elements of $A \times B$ are equal if and only if they have the same first coordinates and the same second coordinates:

$$(a_1, b_1) = (a_2, b_2) \text{ if and only if } a_1 = a_2 \text{ and } b_1 = b_2.$$ 

**Example 12**

Let $A = \{a, b\}$ and $B = \{x, y, z\}$. Then

$$A \times B = \{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}$$

and

$$B \times A = \{(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)\}.$$ 

This example illustrates that, in general, the sets $A \times B$ and $B \times A$ are different.

**Example 13**

The Cartesian plane, in which calculus students sketch curves, is a picture of $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. The adjective Cartesian is derived from Descartes, as Cartesius was Descartes's name in Latin.

**Problem 14.** Let $A$, $B$, and $C$ be sets. Prove that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

**Solution.** We must prove that any element in $A \times (B \cup C)$ is in $(A \times B) \cup (A \times C)$. Since the elements in $A \times (B \cup C)$ are ordered pairs, we begin by letting $(x, y) \in A \times (B \cup C)$ (this is more helpful than starting with "$x \in A \times B$") and ask ourselves what this means. It means that $x$, the first coordinate, is in $A$ and $y$, the second coordinate, is in $B \cup C$. Therefore, $y$ is in either $B$ or $C$. If $y$ is in $B$, then, because $x$ is in $A$, $(x, y) \in A \times B$. If $y$ is in $C$, then, since $x$ is in $A$, $(x, y) \in A \times C$. Thus, $(x, y)$ is either in $A \times B$ or in $A \times C$; thus, $(x, y)$ is in $(A \times B) \cup (A \times C)$, which is what we wanted to show.

Let $A$, $B$, and $C$ be three sets. Prove that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. What can you conclude about the sets $A \times (B \cup C)$ and $(A \times B) \cup (A \times C)$? Why?

Let $A$ and $B$ be nonempty sets. Prove that $A \times B = B \times A$ if and only if $A = B$. Is this true if $A = \emptyset$?

**Answers to Pauses**

4. ($\rightarrow$) Suppose the first statement, $A \cup B = B$, is true. We show $A \subseteq B$. So let $x \in A$. Then $x$ is certainly in $A \cup B$, by the definition of $\cup$. But $A \cup B = B$, so $x \in B$. Thus, $A \subseteq B$.

($\leftarrow$) Conversely, suppose the second statement, $A \subseteq B$, is true. We have to show $A \cup B = B$. To prove the sets $A \cup B$ and $B$ are equal, we have to show each is a subset of the other. First, let $x \in A \cup B$. Then $x$ is either in $A$ or in $B$. If the latter, $x \in B$, and if the former, $x \in B$ because $A$ is a subset of $B$. In either case, $x \in B$. Thus, $A \cup B \subseteq B$. Second, assume $x \in B$. Then $x$ is in $A \cup B$ by definition of $\cup$. So $B \subseteq A \cup B$ and we have equality, as required.

5. $B \setminus A = \{4, 5\}; A^c = (-\infty, -4) \cup (4, \infty)$.

6. An element of $(A \times B) \cup (A \times C)$ is either in $A \times B$ or in $A \times C$; in either case, it's an ordered pair. So we begin by letting $(x, y) \in (A \times B) \cup (A \times C)$ and noting that either $(x, y) \in A \times B$ or $(x, y) \in A \times C$. In the first case, $x$ is in $A$ and $y$ is in $B$; in the second case, $x$ is in $A$ and $y$ is in $C$. In either case, $x$ is in $A$ and $y$ is either in $B$ or in $C$; so $x \in A$ and $y \in B \cup C$. Therefore, $(x, y) \in A \times (B \cup C)$, establishing the required subset relation. The reverse subset relation was established in Problem 14. We conclude that the two sets in question are equal; that is, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

René Descartes (1596–1650), together with Pierre de Fermat, the inventor of analytic geometry, introduced the method of plotting points and graphing functions in $\mathbb{R}^2$ with which we are so familiar today.
7. Suppose that the statement $A \times B = B \times A$ is true. We prove $A = B$. So suppose $x \in A$. Since $B \neq \emptyset$, we can find some $y \in B$. Thus, $(x, y) \in A \times B$. Since $A \times B = B \times A$, $(x, y) \in B \times A$. So $x \in B$, giving us $A \subseteq B$. Similarly, we show that $B \subseteq A$ and conclude $A = B$.

$(\rightarrow)$ On the other hand, if $A = B$ is a true statement, then $A \times B = A \times A = B \times A$.

Finally, if $A = \emptyset$ and $B$ is any nonempty set, then $A \times B = \emptyset = B \times A$, but $A \neq B$. So $A \times B = B \times A$ does not mean $A = B$ in the case $A = \emptyset$.

**True/False Questions**

**(Answers can be found in the back of the book.)**

1. If $A$ and $B$ are sets and $A \neq B$, then $A \cap B \subseteq A \cup B$.
2. If $A$ and $B$ are sets, then $(A \setminus B) \cap (B \setminus A) = \emptyset$.
3. If $A$ and $B$ are sets, then $(A \cup B)^c = A \cap B^c$.
4. If $A$ and $B$ are sets, then $A \setminus B \subseteq A \cap B$.
5. If $A$ and $B$ are sets and $A \neq B$, then $A \cap B \neq \emptyset$.
6. If $A$ and $B$ are nonempty sets, then $A \times B$ is a nonempty set.
7. The name of Augustus De Morgan appears in both Chapter 1 and Chapter 2 of this text.
8. $(A \subseteq B) \to (B^c \subseteq A^c)$.
9. $(B^c \subseteq A^c) \to (A \subseteq B)$.

**Exercises**

The answers to exercises marked [BB] can be found in the Back of the Book.

1. Let $A = \{x \in \mathbb{R} \mid x < 2\}$, $B = \{x \in \mathbb{Z} \mid |x - 2| < 4\}$, and $C = \{x \in \mathbb{R} \mid x^3 - 4x = 0\}$.
   (a) [BB] List the elements in each of these sets.
   (b) Find $A \cup C$, $B \cap C$, $B \setminus C$, $A \oplus B$, $C \times (B \cap C)$, $(A \setminus B) \cap C$, $A \setminus (B \setminus C)$, and $(B \cup C) \cap \emptyset$.
   (c) List the elements in $S = \{(a, b) \in A \times B \mid a = b + 2\}$ and in $T = \{(a, c) \in A \times C \mid a + c = c\}$.

2. Let $S = \{2, 5, \sqrt{2}, 25, \pi, \frac{1}{2}\}$ and $T = \{4, 25, \sqrt{2}, 6, \frac{3}{2}\}$.
   (a) [BB] Find $S \cap T$, $S \cup T$, and $T \cap (S \cap T)$.
   (b) [BB] Find $Z \cup S$, $Z \cap S$, $Z \cup T$, and $Z \cap T$.
   (c) List the elements in each of the sets $Z \cap (S \cup T)$ and $(Z \cap S) \cup (Z \cap T)$. What do you notice?
   (d) List the elements of $Z \cup (S \cap T)$ and list the elements of $(Z \cup S) \cap (Z \cup T)$. What do you notice?

3. Let $A = \{(-1, 2), (4, 5), (0, 0), (6, -5), (5, 1), (4, 3)\}$. List the elements in each of the following sets.
   (a) [BB] $\{a + b \mid (a, b) \in A\}$
   (b) $\{a \mid a > 0 \text{ and } (a, b) \in A \text{ for some } b\}$
   (c) $\{b \mid b = k^2 \text{ for some } k \in Z \text{ and } (a, b) \in A \text{ for some } a\}$

4. List the elements in the sets $A = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \leq b, b \leq 3\}$ and $B = \{\frac{b}{2} \mid a, b \in (-1, 1, 2)\}$.

5. For $A = \{a, b, c, \{a, b\}\}$, find
   (a) [BB] $A \setminus \{a, b\}$
   (b) $\emptyset \cap (\emptyset \setminus A)$
   (c) $\emptyset \setminus \emptyset$
   (d) $\emptyset \setminus A$
   (e) [BB] $\{a, b, c\} \setminus A$
   (f) $(\{a, b, c\} \setminus \{a\}) \setminus A$

6. Find $A^c$ (with respect to $U = \mathbb{R}$) in each of the following cases.
   (a) [BB] $A = (1, \infty) \cup (-\infty, -2)$
   (b) $A = (-3, \infty) \cap (-\infty, 4)$
   (c) $A = \{x \in \mathbb{R} \mid x^2 \leq 1\}$

7. Let $X = \{1, 2, 3, 4\}$, $Y = \{2, 3, 4, 5\}$, and $Z = \{3, 4, 5, 6\}$. List the elements in the indicated sets. (The universal set is the set of integers.)
   (a) [BB] $X \setminus (Y \cap Z)$
   (b) $X^c \cup Y^c$

8. Let $n > 3$ and $A = \{1, 2, 3, \ldots, n\}$.
   (a) [BB] How many subsets of $A$ contain $\{1, 2\}$?
   (b) How many subsets of $B$ of $A$ have the property that $B \cap \{1, 2\} = \emptyset$?
   (c) How many subsets $B$ of $A$ have the property that $B \cup \{1, 2\} = A$?

Explain your answers.

9. [BB] Let $a$ and $b$ be real numbers with $a < b$. Find $(a, b)^c$, $(a, b)$, $(a, \infty)^c$, and $(-\infty, b)^c$. 
10. The universal set for this problem is the set of students attending Miskatonic University. Let
   • $M$ denote the set of math majors
   • $CS$ denote the set of computer science majors
   • $T$ denote the set of students who had a test on Friday
   • $P$ denote those students who ate pizza last Thursday
Using only the set theoretical notation we have introduced in this chapter, rewrite each of the following assertions.
   (a) [BB] Computer science majors had a test on Friday.
   (b) [BB] No math major ate pizza last Thursday.
   (c) Some math majors did not eat pizza last Thursday.
   (d) Those computer science majors who did not have a test on Friday ate pizza on Thursday.
   (e) Math or computer science majors who ate pizza on Thursday did not have a test on Friday.

11. Use the set theoretical notation introduced in this chapter to express the negation of each of statements (a)-(e) in Exercise 10. Do the same for the converse of any statement that is an implication.

12. Let $P$ denote the set of primes and $E$ the set of even integers. As always, $Z$ and $N$ denote the integers and natural numbers, respectively. Find equivalent formulations of each of the following statements using the notation of set theory that has been introduced in this section.
   (a) [BB] There exists an even prime.
   (b) 0 is an integer but not a natural number.
   (c) Every natural number is an integer.
   (d) Not every integer is a natural number.
   (f) $2$ is an even prime.
   (g) $2$ is the only even prime.

13. For $n \in \mathbb{Z}$, let $A_n = \{a \in \mathbb{Z} \mid a \leq n\}$. Find each of the following sets.
   (a) [BB] $A_3 \cup A_{-3}$
   (b) $A_3 \cap A_{-3}$
   (c) $A_3 \setminus (A_{-3})$ (d) $\cap_{i=0}^{n} A_i$

14. [BB] In Fig. 2.1, the region labeled 7 represents the set $C \setminus (A \cup B)$. What set is represented by the region labeled 2? By that labeled 3? By that labeled 4?

15. Let $A = \{1, 2, 4, 5, 6, 9\}$, $B = \{1, 2, 3, 4\}$, and $C = \{5, 6, 7, 8\}$.
   (a) Draw a Venn diagram showing the relationship between these sets. Show which elements are in which region.
   (b) What are the elements in each of the following sets? Show which elements are in each region.
      i. $(A \cup B) \cap C$
      ii. $A \setminus (B \setminus A)$
      iii. $(A \cup B) \setminus (A \cap C)$
      iv. $A \oplus C$
      v. $(A \cap C) \times (A \cup B)$

16. (a) [BB] Suppose $A$ and $B$ are sets such that $A \cap B = A$. What can you conclude? Why?
   (b) Repeat (a) assuming $A \cup B = A$.

17. [BB] Let $n \geq 1$ be a natural number. How many elements are in the set $\{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \leq b \leq n\}$? Explain.

18. Suppose $A$ is a subset of $\mathbb{N} \times \mathbb{N}$ with the properties
   • $(1, 1) \in A$ and
   • if $(a, b) \in A$, then both $(a + 1, b)$ and $(a, b + 1)$ are also in $A$.
   Do you think that $\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m \geq n\}$ is a subset of $A$? Explain. [Hint: A picture of $A$ in the $xy$-plane might help.]

   (a) If $A \cap B \subseteq C$ and $A' \cap B \subseteq C$, prove that $B \subseteq C$.
   (b) [BB] Given that $A \cap B = A \cap C$ and $A' \cap B = A' \cap C$, does it follow that $B = C$? Justify your answer.

20. Let $A$, $B$, and $C$ be sets.
   (a) Find a counterexample to the statement $A \cup (B \cap C) = (A \cup B) \cap C$.
   (b) Without using Venn diagrams, prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

21. Use the first law of De Morgan to prove the second: $(A \cap B)' = A' \cup B'$.

22. [BB] Use the laws of De Morgan and any other set theoretic identities discussed in the text to prove that $(A \setminus B) \setminus C = A \setminus (B \cup C)$ for any sets $A$, $B$, and $C$.

23. Let $A$, $B$, $C$, and $D$ be subsets of a universal set $U$. Use set theoretic identities discussed in the text to simplify the expression $\{A \cup B\} \cap (A' \cup C') \setminus D'$.

24. Let $A$, $B$, and $C$ be subsets of some universal set $U$. Use set theoretic identities discussed in the text to prove that $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$.

25. Suppose $A$, $B$, and $C$ are subsets of some universal set $U$.
   (a) [BB] Generalize the laws of De Morgan by finding equivalent ways to describe the sets $(A \cup B \cup C)'$ and $(A \cap B \cap C)'$.
   (b) Find a way to describe the set $(A \cap (B \setminus C))' \cap A$ without using the symbol $\setminus$ for set complement.

26. Let $A$ and $B$ be sets.
   (a) [BB] Find a necessary and sufficient condition for $A \oplus B = A$.
   (b) Find a necessary and sufficient condition for $A \setminus B = A \cup B$.
   Explain your answers (with Venn diagrams if you wish).

27. Which of the following conditions imply that $B = C$? In each case, either prove or give a counterexample.
   (a) [BB] $A \cup B = A \cup C$
   (b) $A \cap B = A \cap C$
   (c) $A \setminus B = A \setminus C$
   (d) $A \times B = A \times C$

28. True or false? In each case, provide a proof or a counterexample.
   (a) $A \subseteq C$, $B \subseteq D \rightarrow A \times B \subseteq C \times D$.
   (b) $A \subseteq C$, $B \subseteq D \rightarrow A \subseteq C$.
   (c) $A \times B \subseteq C \times D \rightarrow A \subseteq C$ and $B \subseteq D$.
   (d) $A \subseteq C$ and $B \subseteq D$ if and only if $A \times B \subseteq C \times D$.
   (e) [BB] $A \cup B \subseteq A \setminus B \rightarrow A = B$. 

29. Show that sets $A$, $B$, and $C$ are equal, either proving equality or giving a counterexample.
   (a) $A \setminus B = A \setminus C$.
   (b) $A \cap B = A \cap C$.
   (c) $A \cup B = A \cup C$.
   (d) $A \setminus B = A \setminus C$.

30. Let $A$, $B$, and $C$ be sets. Then $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
29. Show that \((A \cap B) \times C = (A \times C) \cap (B \times C)\) for any sets \(A, B,\) and \(C.\)

30. Let \(A, B,\) and \(C\) be arbitrary sets. For each of the following, either prove the given statement is true or exhibit a counterexample to prove it is false.
   (a) \(A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)\)
   (b) \((A \setminus B) \times C = (A \times C) \setminus (B \times C)\)
   (c) \([BB] (A \oplus B) \times C = (A \times C) \oplus (B \times C)\)
   (d) \((A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)\)
   (e) \((A \setminus B) \times (C \setminus D) = (A \times C) \setminus (B \times D)\)

31. Find out what you can about George Boole and write a paragraph or two about him (in good English, of course).

### 2.3 Binary Relations

If \(A\) and \(B\) are sets, remember that the Cartesian product of \(A\) and \(B\) is the set \(A \times B = \{(a, b) \mid a \in A, b \in B}\). There are occasions when we are interested in a certain subset of \(A \times B\). For example, if \(A\) is the set of former major league baseball players and \(B = \mathbb{N} \cup \{0\}\) is the set of nonnegative integers, then we might naturally be interested in

\[ \mathcal{R} = \{(a, b) \mid a \in A, b \in B, \text{ player } a \text{ had } b \text{ career home runs}\}. \]

For example, (Hank Aaron, 755) and (Mickey Mantle, 536) are elements of \(\mathcal{R}\).

#### 2.3.1 Definitions

Let \(A\) and \(B\) denote sets. A **binary relation from \(A\) to \(B\)** is a subset of \(A \times B\). A **binary relation on \(A\)** is a subset of \(A \times A\).

#### 2.3.2 Remark

When \(\mathcal{R}\) is a binary relation from \(A\) to \(B\) and the pair \((a, b)\) is in \(\mathcal{R}\), we naturally write \((a, b) \in \mathcal{R}\), though the reader should be aware that other authors prefer the notation \(a \mathcal{R} b\).

The empty set and the entire Cartesian product \(A \times B\) are always binary relations from \(A\) to \(B\), although these are generally not as interesting as certain nonempty proper subsets of \(A \times B\).

**Example 15**

- If \(A\) is the set of students who were registered at the University of Toronto during the Fall 2001 semester and \(B\) is the set \{History, Mathematics, English, Biology\}, then \(\mathcal{R} = \{(a, b) \mid a \in A \text{ is enrolled in a course in subject } b\}\) is a binary relation from \(A\) to \(B\).
- Let \(A\) be the set of surnames of people listed in the Seattle telephone directory. Then \(\mathcal{R} = \{(a, n) \mid a \text{ appears on page } n\}\) is a binary relation from \(A\) to the set \(\mathbb{N}\) of natural numbers.
- \(\{(a, b) \mid a, b \in \mathbb{N}, \frac{a}{b} \text{ is an integer}\}\) and \(\{(a, b) \mid a, b \in \mathbb{N}, a - b = 2\}\) are binary relations on \(\mathbb{N}\).
- \(\{(x, y) \mid y = x^2\}\) is a binary relation on \(\mathcal{R}\) whose graph the reader should recognize.

What is the common name for this graph?

Our primary intent in this section is to identify special properties of binary relations on a set, so, henceforth, all binary relations will be subsets of \(A \times A\) for some set \(A\).

#### 2.3.3 Definition

A binary relation \(\mathcal{R}\) on a set \(A\) is **reflexive** if and only if \((a, a) \in \mathcal{R}\) for all \(a \in A\).