1. Prove the following proposition directly from this definition:

**Definition:** If \( x \in \mathbb{R}, x > 0 \), then \( \log x = \int_1^x \frac{dt}{t} \).

**Proposition:** The function \( \log : \{ x \in \mathbb{R} : x > 0 \} \to \mathbb{R} \) is differentiable with \( \frac{d\log x}{dx} = \frac{1}{x} \), it is strictly increasing, assumes all values in \( \mathbb{R} \), and satisfies the rules:

(a) \( \log xy = \log x + \log y \) if \( x, y > 0 \),
(b) \( \log \frac{x}{y} = \log x - \log y \) if \( x, y > 0 \),
(c) \( \log x^n = n \log x \) if \( x > 0, n \) an integer.

2. Prove the following proposition directly from this definition:

**Definition:** \( \exp \) is the inverse function of \( \log \), that is \( \exp(x) = y \) means \( x = \log y \).

**Proposition:** The function \( \exp : \mathbb{R} \to \{ x \in \mathbb{R} : x > 0 \} \) is differentiable, with \( \frac{d\exp(x)}{dx} = \exp(x) \). It is strictly increasing, assumes all positive values, and satisfies the rules:

(a) \( \exp(x) \cdot \exp(y) = \exp(x + y) \) if \( x, y \in \mathbb{R} \),
(b) \( \frac{\exp(x)}{\exp(y)} = \exp(x - y) \) if \( x, y \in \mathbb{R} \),
(c) \( \exp(nx) = (\exp(x))^n \) if \( x \in \mathbb{R}, n \) an integer.

3. Prove the following proposition directly from this definition:

**Definition:** If \( x, n \in \mathbb{R}, x > 0 \), then \( x^n = \exp(n \log x) \).

**Proposition:** For \( x, y, n, m \in \mathbb{R}, x, y > 0 \), we have:

(a) \( x^n \cdot x^m = x^{n+m} \),
(b) \( \frac{x^n}{x^m} = x^{n-m} \),
(c) \( (x^n)^m = x^{nm} \),
(d) \( (xy)^n = x^ny^n \),
(e) \( \frac{d}{dx} x^n = nx^{n-1} \).

4. **Definition:** \( e = \exp(1) \).

Can the exponential function \( e^x \) be derived from scratch, starting with the function \( x^n \) for fixed \( n \in \mathbb{N} \)?

(a) Does any positive number have a unique positive \( n \)th root?
(b) Can rational powers \( x^{m/n} \) be defined appropriately?
(c) Does this extend to irrational powers?
(d) Based on this, for fixed \( a \in \mathbb{R} \), is the function \( a^x \) differentiable?
(e) Does there exist some \( a \in \mathbb{R} \) such that \( \frac{da^x}{dx} = a^x \)?