There are 10 problems on 2 pages of this assignment.

Let $X$ and $Y$ be sets with $A \subseteq X$ and $B \subseteq Y$. Let $f : X \to Y$ be a function. The image of $A \subseteq X$ is the subset of $Y$ that consists of the images of the elements of $A$, denoted $f(A)$. Hence, $f(A) = \{f(a) \mid a \in A\} \subseteq Y$. The inverse image of $B \subseteq Y$ is the subset of $X$ containing all pre-images of all elements of $B$, denoted $f^{-1}(B)$. Hence, $f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X$.

1. Let $A$ be a nonempty set and let $f : A \to A$ be a function. Prove that if $f \circ f = i_A$, then $f$ is bijective.

2. Let $A$ and $B$ be nonempty sets. Prove that if $f : A \to B$, then $f \circ i_A = f$ and $i_B \circ f = f$.

3. For nonempty sets $A$ and $B$ and functions $f : A \to B$ and $g : B \to A$, suppose that $g \circ f = i_A$.
   (a) Prove that $f$ is one-to-one and $g$ is onto.
   (b) Show that $f$ need not be onto.
   (c) Show that $g$ need not be one-to-one.
   (d) Prove that if $f$ is onto, then $g$ is one-to-one.
   (e) Prove that if $g$ is on-to-one, then $f$ is onto.

4. Let $A$, $B$, and $C$ be nonempty sets. Suppose that $f$ is a function from $A$ to $B$ and $g$ is a function from $B$ to $C$.
   (a) Prove that if $f$ and $g$ are one-to-one, then so is $g \circ f$.
   (b) Prove that if $g \circ f$ is one-to-one, then so is $f$.
   (c) Prove that if $f$ is onto and $g \circ f$ is one-to-one, then $g$ is one-to-one.
   (d) Give an example where $g \circ f$ is one-to-one, but $g$ is not.
   (e) Prove that if $f$ and $g$ are onto, then so is $g \circ f$.
   (f) Prove that if $g \circ f$ is onto, then so is $g$.
   (g) Prove that if $g$ is one-to-one and $g \circ f$ is onto, then $f$ is onto.
   (h) Give an example where $g \circ f$ is onto, but $f$ is not.

5. Prove that a function $f : A \to B$ is a bijection if and only if there exists $g : B \to A$ with $g \circ f = i_A$ and $f \circ g = i_B$.

6. Let $S$ be a subset of a universal set $\mathcal{U}$. The characteristic function $f_S$ of $S$ is the function from $\mathcal{U}$ to the set $\{0, 1\}$ such that $f_S(x) = 1$ if $x \in S$ and $f_S(x) = 0$ if $x \notin S$.

Let $A$ and $B$ be subsets of $\mathcal{U}$. Show that for all $x$
   (a) $f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$
   (b) $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$
   (c) $f_{\overline{A}}(x) = 1 - f_A(x)$
7. Let $X$, $Y$, and $Z$ be sets with $A, B \subseteq X$ and $C, D \subseteq Y$. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove the following statements.

(a) If $A \subseteq B$, then $f(A) \subseteq f(B)$.
(b) If $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$.
(c) $f \circ f^{-1}$ is the identity on $\mathcal{P}(Y)$ if $f$ is onto.
(d) $f^{-1} \circ f$ is the identity on $\mathcal{P}(X)$ if $f$ is one-to-one.

8. Let $X$, $Y$, and $Z$ be sets with $A, B \subseteq X$ and $C, D \subseteq Y$. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove the following statements.

(a) $f(A \cup B) = f(A) \cup f(B)$
(b) $f(A \cap B) \subseteq f(A) \cap f(B)$
(c) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
(d) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
(e) $f^{-1}(f(A)) \supseteq A$
(f) $f(f^{-1}(C)) \subseteq C$.

9. Let $X$, $Y$, and $Z$ be sets with $A, B \subseteq X$ and $C, D \subseteq Y$. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove the following statements.

(a) $f$ is one-to-one if and only if $f^{-1}(f(A)) = A$ for all $A \subseteq X$.
(b) $f$ is onto if and only if $f(f^{-1}(C)) = C$ for all $C \subseteq Y$.
(c) $f$ is one-to-one if and only if $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq X$.

10. Let $f : X \to Y$ be a given function. Complement sets are taken within $X$ for subsets of $X$ and within $Y$ for subsets of $Y$.

(a) Prove that $f^{-1}(B) = f^{-1}(\overline{B})$ for all $B \subseteq Y$.
(b) Prove that $f(A) \supseteq f(\overline{A})$ for all $A \subseteq X$ if and only if $f$ is onto.
(c) Prove that $f(\overline{A}) \subseteq f(A)$ for all $A \subseteq X$ if and only if $f$ is one-to-one.