Homework 5

There are 10 problems on 2 pages of this homework assignment.

1. Let \( S = \{x_1, x_2, x_3\} \). Let \( i, f, g \in A(S) = S_3 \) be the functions \( i : x_1 \rightarrow x_1, x_2 \rightarrow x_2, x_3 \rightarrow x_3, \)
   \( f : x_1 \rightarrow x_2, x_2 \rightarrow x_3, x_3 \rightarrow x_1, \) and \( g : x_1 \rightarrow x_2, x_2 \rightarrow x_1, x_3 \rightarrow x_3. \)
   
   (a) Write out all elements of \( S_3 \) in terms of \( i, f, \) and \( g \) in the simplest possible way.
   
   (b) Write down the product table of \( S_3 \). For convenience, write \( f \circ g \) as \( fg \), etc.
   
   (c) Prove that \( (fg)^2 \neq f^2g^2. \)
   
   (d) Prove that \( gf = f^{-1}g \) but \( f \neq f^{-1}. \)
   
   (e) How would your answers above change if \( S = \{a, b, c\} \)?
   
   (f) If \( h \in S_3 \), prove that \( h^6 = i. \)

2. Let \( S \) be the \( x, y \)-plane and consider \( f, g \in A(S) \) defined by
   \( f(x, y) = (-x, y) \) and \( g(x, y) = (-y, x) \). Define \( G = \{f^ig^j \mid i = 0, 1; j = 0, 1, 2, 3\} \)
   with the product \( * \) of function composition.
   
   (a) Give a geometric description of \( f \) and \( g \) in terms of their action on \( S. \)

   (b) How many elements does \( G \) have?

   (c) Prove \( f^2 = g^4 = i_S. \)

   (d) Prove that \( f * g \neq g * f. \)

   (e) Prove that \( g * f = f * g^{-1}. \)

   (f) Find a formula for \( (f^ig^j) * (f^sg^t) = f^ag^b \) that expresses \( a, b \) in terms of integers \( i, j, s, \)
   and \( t. \)

   (g) Write down the product table of \( G. \)

   (h) Prove that \( G \) is a nonabelian group of order 8. This group is called the dihedral group of
   order 8.

3. Let \( S \) be a nonempty and finite set.
   
   (a) Prove that if \( f \) is a mapping of \( S \) onto itself, then \( f \) is 1-1.
   
   (b) Prove that if \( f \) is a 1-1 mapping of \( S \) into itself, then \( f \) is onto.
   
   (c) Prove that the first statement is not true if \( S \) is infinite.

   (d) Prove that the second statement is not true if \( S \) is infinite.

   (e) Prove that if \( f \) is a 1-1 mapping of \( S \) onto \( S, \) then for some integer \( n > 0, f^n = i_S. \)
   (Recall that \( f^n \) is \( f \) composed with itself \( n \) times, and \( i_S \) is the identity function on \( S. \))

   (f) Suppose that \( |S| = m. \) Find an \( n \) (in terms of \( m \)) for the previous problem that works
   for all \( f \in S_m. \)
4. Let \( f, g \in A(S) \). (Recall that \( A(S) \) is the set of all 1-1 mappings from a nonempty \( S \) onto itself.)
   
   (a) Prove that \((fg)^2 = f^2g^2\) if and only if \( fg = gf \).
   
   (b) Prove that \((fg)^{-1} = g^{-1}f^{-1}\).
   
   (c) Prove that if \( fg = gf \), then \((fg)^{-1} = f^{-1}g^{-1}\).
   
   (d) Prove that if \(|S| \geq 3\), then there exists \( f, g \in A(S) \) such that \( fg \neq gf \).

5. Let \( Z_n^* = \mathbb{Z}_n \setminus \{0\} \). Prove, from the definitions, that \( Z_n^* \) is a group under multiplication if and only if \( n \geq 2 \) is a prime. Hint: see ICA 10, problems # 2 – 9.

6. Let \( G \) be a group with identity \( e \). For \( a \in G \), the inverse of \( a \) is denoted \( a^{-1} \).

   (a) Prove that \((a^{-1})^{-1} = a\) for all \( a \in G \).
   
   (b) Prove that the identity \( e \in G \) is unique.
   
   (c) Prove that \((ab)^{-1} = b^{-1}a^{-1}\) for all \( a, b \in G \).
   
   (d) Prove that for all \( a \in G \), \( a^{-1} \) is unique.
   
   (e) Prove that if \( ab = ac \), then \( b = c \) for all \( a, b, c \in G \).
   
   (f) Prove that if \( ba = ca \), then \( b = c \) for all \( a, b, c \in G \).

7. Let \( G \) be a finite group with identity \( e \in G \).

   (a) Prove that for all \( a \in G \) there exists \( n \in \mathbb{N} \) with \( n > 0 \) such that \( a^n = e \).
   
   (b) Prove that there exists \( m \in \mathbb{N} \) with \( m > 0 \) such that \( a^m = e \) for all \( a \in G \).

8. Prove that a group of order 5 or less is abelian.

9. Let \( G \) be a group.

   (a) Let \( a, b \in G \). Find an expression for \((ab)^{-1}\) in terms of \( a^{-1} \) and \( b^{-1} \). Prove your answer is correct.

   (b) Suppose that \( a = a^{-1} \) for every \( a \in G \). Prove that \( G \) is abelian.

10. If \( G \) is a finite group of even order, show that there must be an element \( a \neq e \) such that \( a = a^{-1} \). Hint: What does \((a^{-1})^{-1}\) equal?