There are 12 problems on 2 pages of this homework assignment.

1. Prove the following properties of the absolute value.
   (a) For all \(a, b \in \mathbb{R}\), \(|a + b| \leq |a| + |b|\).
   (b) For all \(a, b \in \mathbb{R}\), \(|b| \leq a\) if and only if \(-a \leq b \leq a\).
   (c) For all \(a, b \in \mathbb{R}\), \(||a| - |b|| \leq |a - b|\).
   (d) If \(a, x, \epsilon \in \mathbb{R}\), then \(|x - a| < \epsilon\) if and only if \(a - \epsilon < x < a + \epsilon\).
   (e) If \(a, b, x, y \in \mathbb{R}\) and \(a < x < b\), \(a < y < b\), then \(|y - x| < b - a\).

2. Let \(a_1, a_2, \ldots, a_n \in \mathbb{R}\) for \(n \in \mathbb{Z}\) with \(n \geq 3\). Prove that \(|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|\).

3. Consider each sequence \((s_n)_{n \in \mathbb{N}}\) and possibly limit \(s\) below. Let \(\epsilon_i = 10^{-i}\) for \(i = 0, 1, 2, 3, 4\). Determine if there exists an integer \(N_i\) such that \(|s_n - s| < \epsilon_i\) for all \(n > N_i\). If so, state the \(N_i\) for each \(i \in \{0, 1, 2, 3, 4\}\). If not, explain why not.
   (a) \(s_n = 1/n^2\) and \(s = 0\).
   (b) \(s_n = (3n + 1)/(7n - 4)\) and \(s = 3/7\).
   (c) \(s_n = [(n + 6)/(n^2 - 6)]\) and \(s = 0\).
   (d) \(s_n = (-1)^n\) and \(n = 1\).

4. Let \(a, b \in \mathbb{R}\). Prove that if \(a \leq b_1\) for all real numbers \(b_1 > b\), then \(a \leq b\).

5. Find the maximum and minimum of the following sets, if they exist. Justify your answers.
   (a) \([a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}\) for some \(a, b \in \mathbb{R}\). The closed interval from \(a\) to \(b\).
   (b) \((a, b) = \{x \in \mathbb{R} \mid a < x < b\}\) for some \(a, b \in \mathbb{R}\). The open interval from \(a\) to \(b\).
   (c) \([a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}\) for some \(a, b \in \mathbb{R}\). One half-open interval from \(a\) to \(b\).
   (d) \((a, b) = \{x \in \mathbb{R} \mid a < x \leq b\}\) for some \(a, b \in \mathbb{R}\). The other half-open interval from \(a\) to \(b\).
   (e) \(\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}, \emptyset\).
   (f) \(\{r \in \mathbb{Q} \mid 0 < r < \sqrt{2}\}\).
   (g) \(\{n(-1)^n \mid n \in \mathbb{N}\}\) where \(\mathbb{N} = \{1, 2, 3, \ldots\}\).

6. Prove that for all \(a, b \in \mathbb{R}\),
   \[
   \max\{a, b\} = \frac{a + b + |a - b|}{2} \quad \text{and} \quad \min\{a, b\} = -\max\{-a, -b\} = \frac{a + b - |a - b|}{2}.
   \]

7. Find the supremum (least upper bound) and infimum (greatest lower bound) for the sets in Problem 5, if they exist. Use the notation \(\sup S\) and \(\inf S\) for the set \(S\). Justify your answers.
8. The *completeness axiom* for the real numbers says: A nonempty subset $S$ of $\mathbb{R}$ that is bounded from above has a least upper bound. In other words, for $S \subseteq \mathbb{R}$, if $S \neq \emptyset$ and there exists $b \in \mathbb{R}$ such that $t \leq b$ for all $t \in S$, then $\sup S$ exists and is a real number.

Let $S \subseteq \mathbb{R}$ be nonempty. Suppose $S$ is bounded from below. Let $-S = \{-s \mid s \in S\}$.

(a) Prove that $\sup(-S)$ exists.
(b) Let $s_0 = \sup(-S)$. Prove that $-s_0 \leq s$ for all $s \in S$.
(c) Let $s_0 = \sup(-S)$. Prove that if $t \leq s$ for all $s \in S$, then $t \leq -s_0$.
(d) Prove $\inf S = -\sup(-S)$.
(e) Conclude that every nonempty set of real numbers that is bounded from below has a greatest lower bound.

9. Let $S$ be a nonempty subset of $\mathbb{R}$ that is bounded from above. Prove that if $\sup S$ belongs to $S$, then $\sup S = \max S$.

10. Let $S$ be a nonempty bounded (that is, bounded from above and bounded from below) subset of $\mathbb{R}$. *You may assume that Problem 8e is true.*

   (a) Prove that $\inf S \leq \sup S$.
   (b) What can be said about $S$ if $\inf S = \sup S$? Justify your answer.

11. Let $S$ and $T$ be nonempty bounded subsets of $\mathbb{R}$.

   (a) Prove that if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.
   (b) Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$. *Do not assume $S \subseteq T$.*

12. Prove the following using the completeness axiom and the ordering of real numbers.

   (a) For all $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}$ such that $n > x$.
   (b) For all $\epsilon \in R$ with $\epsilon > 0$ there exists $n \in \mathbb{Z}$ such that $1/n < \epsilon$.
   (c) For all $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}$ such that $n \leq x < n + 1$.
   (d) For all $x \in \mathbb{R}$ and $N \in \mathbb{Z}$ with $N > 0$, there exists $n \in \mathbb{Z}$ such that $n/N \leq x < (n+1)/N$.
   (e) For all $x, \epsilon \in R$ with $\epsilon > 0$, there exists a rational number $q \in \mathbb{Q}$ such that $|x - q| < \epsilon$. 