1. If a group $G$ has no proper subgroups, prove that $G$ is cyclic of order $p$, where $p$ is a prime.

2. Suppose that $A$ and $B$ are finite subgroups, of orders $m$ and $n$, respectively, of the abelian group $G$.
   
   (a) Prove that $AB$ is a subgroup of order $mn$ if $m$ and $n$ are relatively prime.
   
   (b) What is the order of $AB$ if $m$ and $n$ are not relatively prime?

3. Let $G$ be a group and $H$ a subgroup of $G$. Let $Hx = \{ hx \mid h \in H \}$.
   
   (a) For all $a, b \in G$, prove that either $Ha = Hb$ or $Ha \cap Hb = \emptyset$.
   
   (b) Suppose $H$ is finite. Prove that $Ha$ and $Hb$ have the same number of elements for all $a, b \in G$.
   
   (c) If $H$ is finite, what is $|Ha|$?

4. Let $G$ be a set with an operation $\ast$ such that:
   
   • $G$ is closed under $\ast$.
   • $\ast$ is associative.
   • There exists an element $e \in G$ such that $e \ast x = x$ for all $x \in G$.
   • Given $x \in G$, there exists a $y \in G$ such that $y \ast x = e$.

   Prove that $G$ is a group.

5. (a) Use induction to prove that every nonempty and finite subset of a lattice has a least upper bound and a greatest lower bound.
   
   (b) Prove the statement is false without the finiteness condition.
   
   (c) Prove that the statement is false even if the set is bounded from above and from below.

6. The Harmonic numbers $H_k$ for integer $k \geq 1$ are defined by $H_k = \sum_{i=1}^{k} \frac{1}{i}$.
   
   (a) Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for all nonnegative integers $n$.
   
   (b) Prove that $H_{2^n} \leq 1 + n$ for all nonnegative integers $n$.
   
   (c) Prove that $H_1 + H_2 + \ldots + H_n = (n + 1)H_n - n$ for all integers $n \geq 1$.