1. For each of the four group axioms, give an example of a set with a binary product for which the axiom fails.

2. Let $G$ be a group. (When not necessary for clarity the product $a * b$ is written as $ab$ for $a, b \in G$ and the group is denoted as simply $G$ instead of $(G, *)$.)

   (a) Prove that if $ab = ac$, then $b = c$ for all $a, b, c \in G$.
   (b) Prove that if $ba = ca$, then $b = c$ for all $a, b, c \in G$.
   (c) Based on these results, what conclusions can be drawn about the product table of a group? How many times does each element appear in a given row? In a given column?

3. Let $G$ be a group.

   (a) Prove that the identity $e \in G$ is unique.
   (b) Prove that for all $a \in G$, $a^{-1}$ is unique.
   (c) Prove that $(a^{-1})^{-1} = a$ for all $a \in G$.
   (d) Prove that $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$.

4. Let $S$ be a nonempty set. The set of all 1-1 mappings from $S$ onto itself is denoted $A(S)$. For $f \in A(S)$ we define $f^0 = i_S$, $f^1 = f$, and $f^n = f \circ f^{n-1}$ for $n \in \mathbb{N}$ with $n \geq 2$. We also define $f^{-n} = (f^{-1})^n$ for $n \in \mathbb{N}$. It follows that $f^n f^m = f^{n+m}$ and $(f^n)^m = f^{nm}$ for $n, m \in \mathbb{N}$.

   Prove, from the definitions, that $A(S)$ is a group under the operation of function composition.

5. Let $Z_n^* = \mathbb{Z}_n \setminus \{[0]\}$. Prove that $Z_n^*$ is a group under multiplication if and only if $n \geq 2$ is a prime.