Solutions are to be written on the board.

1. Prove the following properties of $\mathbb{R}$ from the definition of $(\mathbb{R}, +)$ and $(\mathbb{R}^*, \cdot)$ as abelian groups where multiplication distributes over addition: $a(b + c) = ab + bc$ for all $a, b, c \in \mathbb{R}$.

Let $a, b, c \in \mathbb{R}$.

(a) $a \cdot 0 = 0$.

(b) $(-a)b = -ab$.

(c) $(-a)(-b) = ab$.

(d) If $ab = 0$, then either $a = 0$ or $b = 0$.

(e) $-a = (-1)a$.

It will be very helpful to recall that you have already proved (so could easily reprove):

- $a + c = b + c$ implies $a = b$. Likewise, if $ac = bc$ and $c \neq 0$, then $a = b$. Hence the equation $x + a = b$ has a unique solution, as does $ax = b$ when $a \neq 0$.
- $-(-a) = a$ and $(a^{-1})^{-1} = a$.
- $-(a + b) = (-a) + (-b)$ and $(ab)^{-1} = a^{-1}b^{-1}$.

2. Prove the following properties of $\mathbb{R}$ from the group definitions above, along with the total ordering of $\mathbb{R}$ under the relation $\leq$ with the additional properties that, for all $a, b, c \in \mathbb{R}$, if $a \leq b$, then $a + c \leq b + c$ and if $a \leq b$ with $0 \leq c$, then $ac \leq bc$.

Let $a, b, c \in \mathbb{R}$.

(a) If $a \leq b$, then $-b \leq -a$.

(b) If $a \leq b$ and $c \leq 0$, then $bc \leq ac$.

(c) If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

(d) $0 \leq a^2$.

(e) $0 < 1$.

(f) If $0 < a$, then $0 < a^{-1}$.

(g) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.