CHAPTER I. INTEGRAL EQUATIONS
A COMPRENDIUM OF PROBLEMS

I. Consider the problem

\[ y(x) = \int_{-1}^{1} \left( \frac{1}{2} + x t \right) y(t) \, dt + f(x). \]

(a) Explain how you know this problem is in the second alternative.

ans: \( y(x) = c \) is a non-trivial solution to the non-homogeneous problem.

(b) Find linearly independent solutions for the equation \( y = K(y) \).

(c) Let \( f_1(x) = 3x - 1 \) and \( f_2(x) = 3x^2 - 1 \). For one of these there is a solution to the equation \( y = K(y) + f \), for the other there is not. Which has a solution?

ans: \( 3x^2 - 1 \).

(d) For the \( f \) for which there is a solution, find two.

ans: \( 3x^2 - 1 + 7 \) and \( 3x^2 - 1 + 11 \).

II Consider the problem

\[ y(x) = \int_{0}^{1} x^2 t^2 y(t) \, dt + f(x). \]

(a) Show that the associated \( K \) is small in both senses of this section.

(b) Compute \( \phi_2 \) where \( f(x) = 1 \).

ans: \( \frac{2}{5}x^2 + 1 \)

(c) Give an estimate for how much \( \phi_2 \) differs from the solution \( y \) of \( y = K(y) + f \).

ans: error \( \leq \frac{1}{2425^2} \)

(d) Using the kernel \( k \) for \( K \), compute the kernel \( k_2 \) for \( K^2 \) and \( k_3 \) for \( K^3 \).

ans: \( k_2(x,t) = x^2 t^2 / 5 \).

(e) Compute the kernel for the resolvent of this problem.

ans: \( r(x,t) = \frac{5x^2 t^2}{4} \)

(f) What is the solution for \( y = K y + f \) in case \( f(x) = 1 \).

ans: \( y(x) = 1 + \frac{5}{12}x^2 \)

III. Consider the problem

\[ y(x) = \int_{0}^{1} x t^3 y(t) \, dt + x^2. \]

(a) Compute the associated approximations \( \phi_0 \), \( \phi_1 \), \( \phi_2 \) and \( \phi_3 \).

ans: \( \phi_1(x) = x^2 + x/6 \)
(b) Give an estimate for how much $\phi_3$ differs from the solution.
(c) Give the kernel for the resolvent of this problem.
\[ \text{ans: } r(x,t) = \frac{5xt^3}{4} \]
(d) Using the resolvent, give the solution to this problem.
\[ \text{ans: } y(x) = x^2 + \frac{5x}{24} \]
(e) Using the fact that the kernel of the problem separates, solve the equation.

IV. Suppose that \( K(x,t) = \begin{cases} 1-t & \text{if } x < t \\ 1-x & \text{if } t < x \end{cases} \)
(a) Show that \( \left| K(x,t) \right| dt < 1 \) for all \( x \) in \([0,1]\).
(b) Solve the problem \( y = K[y] + 1 \). \[ \text{ans: } y(x) = \frac{\cos(x)}{\cos(1)} \]

V. a. Find a nontrivial solution for \( y = K[y] \) in \( L^2[0,1] \) where
\[ K(x,t) = 1 + \cos(\pi x) \cos(\pi t) \]
b. Find a nontrivial solution for \( z = K^*[z] \).
c. What condition must hold on \( f \) in order that
\[ y = K[y] + f \]
shall have a solution. Does \( f(x) = 3x^2 \) meet this condition.
\[ \text{ans: } \text{Constant functions are nontrivial solutions for both equations and the equation of IV(c) has a solution provided} \]
\[ \int_0^1 f(t) dt = 0. \]
The function \( 3x^2 \) does not meet this condition.