CHAPTER I. INTEGRAL EQUATIONS

SECTION 1.1 GEOMETRY AND ONE TYPE OF LINEAR FUNCTION

Most often, freshman and sophomore mathematics is a study of the calculus of $\mathbb{R}^n$, likely with $n = 1, 2, \text{ or } 3$. Instead of working in the space $\mathbb{R}^n$, we now work in a space of functions on a finite interval. Most often, we will take that interval to be $[0,1]$. Of course, we will not work in the class of all functions on $[0,1]$; rather we ask that the linear space should consist of functions $f$ for which

$$
\int_0^1 |f(x)|^2 \, dx < \infty.
$$

Then, we have an inner product space as we did in the finite dimensional calculus. This space is called $L^2([0,1])$. The dot product of two functions is given by

$$
<f, g> = \int_0^1 f(x) g(x) \, dx
$$

and the norm of $f$ is defined in terms of the dot product:

$$
||f||^2 = \int_0^1 |f(x)|^2 \, dx.
$$

It does not seem appropriate to study in detail the nature of $L^2[0,1]$ at this time. Rather, suffice it to say that the space is large enough to contain all continuous functions - even functions which are continuous except at a finite number of places. The interested student can find what $L^2[0,1]$ is by looking in standard books in Real Analysis.

Having an inner product space, we can now decide if $f$ and $g$ in the space are perpendicular. The distance and the angle between $f$ and $g$ are given by the same formulas as we understood from the finite dimensional calculus: the distance from $f$ to $g$ is $||f - g||$ and the angle $\alpha$ between $f$ and $g$ satisfies

$$
\cos(\alpha) = \frac{<f, g>}{||f|| \cdot ||g||}
$$

provided neither $f$ nor $g$ is zero.

Suppose $\{f_p\}_{p=1}^\infty$ is a sequence of functions in $L^2([0,1])$. It is
valuable to consider the possible meanings for \( \lim_p f_p(x) = g(x) \). There are at least three meanings.

The sequence \( f_p \) converges point-wise to \( g \) at each \( x \) in \([0,1]\) provided that for each \( x \) in \([0,1]\),

\[
\lim_p f_p(x) = g(x).
\]

The sequence converges to \( g \) uniformly on \([0,1]\) provided that

\[
\lim_p \sup_x |f_p(x) - g(x)| = 0.
\]

And, the sequence converges to \( g \) in norm if

\[
\lim_p ||f_p - g|| = 0.
\]

An understanding of these three modes of convergence should be sought. These are ideas that re-occur in mathematics. In class, we will give examples to contrast these methods of convergence.

A type of integral equation will be studied in this section. For example, given a function called the kernel

\[
K: [0,1] \times [0,1] \rightarrow \mathbb{R}
\]

and a function \( f: [0,1] \rightarrow \mathbb{R} \), we seek a function \( y \) such that for each \( x \) in \([0,1]\),

\[
y(x) = \int_0^1 K(x,t) y(t) \, dt + f(x).
\]

Such equations are called Fredholm equations of the second kind. An equation of the form

\[
0 = \int_0^1 K(x,t) y(t) \, dt + f(x)
\]

is a Fredholm equation of the first kind.

The requirements in this section on \( K \) and \( f \) will be that

\[
\int_0^1 \int_0^1 |K(x,t)|^2 \, dx \, dt < \infty \quad \text{and} \quad \int_0^1 |f(x)|^2 \, dx < \infty.
\]

These requirements are met if \( K \) and \( f \) are continuous.
For simplicity, we denote by \( K \) the linear function given by

\[
K(y)(x) = \int_0^1 K(x,t) y(t) \, dt.
\]

Note that \( K \) has a domain large enough to contain all functions \( y \) which are continuous on \([0,1]\). Also, if \( y \) is continuous then \( K(y) \) is a function and its value at \( x \) is denoted \( K(y)(x) \). In spoken conversation, it is not so easy to distinguish the number valued function \( K \) and the function valued \( K \). The bold character will be used in these notes to denoted the latter.

It is well to note the resemblence of this function \( K \) to the multiplication of a matrix \( A \) by a vector \( u \):

\[
A(u)(p) = \sum_{q=1}^{\infty} A(p,q) u(q).
\]

This formula has the same form as that for \( K \) given above. The analogy should be instructive.

In order to understand \( K^* \), one must consider \( \langle K(f), g \rangle \) and seek \( K^* \) such that \( \langle Kf, g \rangle = \langle f, K^*g \rangle \).

\[
\langle K(f), g \rangle = \int_0^1 K(f)(x) \, g(x) \, dx
\]

\[
= \int_0^1 \int_0^1 K(x,t) f(t) \, g(x) \, dt \, dx.
\]

An examination of these last equations leads one to guess that \( K^* \) is given by

\[
K^*(g)(t) = \int_0^1 K(x,t) \, g(x) \, dx,
\]

or, keeping \( t \) as the variable of integration,

\[
K^*(g)(x) = \int_0^1 K(t,x) \, g(t) \, dt,
\]

Those last equations verified that

\[
\langle K(f), g \rangle = \langle f, K^*(g) \rangle.
\]
Care had to be taken to watch whether the "variable of integration" is t or x in the integrals involved.

In summary, if $K$ is the kernel associated with the linear operator $K$, then the kernel associated with $K^*$ is given by $K^*(x,y) = K(y,x)$. It is of value to compare how to get $K^*$ from $K$ with the process of how to get $A^*$ from $A$:

$$A^*_{p,q} = A_{q,p}.$$  

Consistent with the rather standard notation we have adopted above, it is clear that a briefer representation of the equation

$$\frac{1}{y(x)} = \int_0^1 K(x,t) y(t) \, dt + f(x)$$

is the concise equation $y = K(y) + f$, or $(1 - K)y = f$.

**EXAMPLE**: Suppose that

$$K(x,t) = \begin{cases} (x-t)^2 & \text{if } 0 < x < t < 1 \\ 0 & \text{if } 0 < t < x < 1 \end{cases}.$$  

To get $K^*$, let's use other letters for the argument of $K^*$ and $K$ to avoid confusion. Suppose that $0 < u < v < 1$. Then, $K^*(u,v) = K(v,u) = 0$. In a similar manner, $K^*(u,v) = (u-v)^2$ if $0 < v < u < 1$. Note that $K^*$ is not $K$.

$$K^*(x,t) = \begin{cases} 0 & \text{if } 0 < x < t < 1 \\ (x-t)^2 & \text{if } 0 < t < x < 1 \end{cases}.$$  

The discussion of this example has been algebraic to this point. Consider this geometric notion that is suggested by the alternate name for "self-adjoint", namely, some call $K$ "symmetric" if $K(x,t) = K(t,x)$. The geometric name suggests a picture and the picture is the graph of $K$. The $K$ of this example is not symmetric in $x$ and $t$. Its graph is not symmetric about the line $x = t$. The function $K$ is different from the function $K^*$.  

EXERCISE 2.1:

1. (a) Find the distance from $\sin(\pi x)$ to $\cos(\pi x)$ in $L^2[0,1]$ and $L^2[-1,1]$.  
   Ans: $1, \sqrt{2}$

(b) Find the angle between $\sin(\pi x)$ and $\cos(\pi x)$ in $L^2[0,1]$ and $L^2[-1,1]$.  
   Ans: $\pi/2, \pi/2$.

2. Repeat 1. (a) and (b) for $x$ and $x^2$.  
   Ans: $1/\sqrt{30}$, $4/\sqrt{15}$, $\arccos(\sqrt{15}/4)$, $\pi/2$.

3. Suppose $K(x,t) = 1 + 2x^2t^2$ on $[0,1] \times [0,1]$ and $y(x) = 3 - x$. Compute $K(y)$ and $K^*(y)$.  
   Ans: $(5+3x)/2$, $(15+14x^2)/6$.

4. Suppose $K(x,t) = \begin{cases} 
  x \ t \ 	ext{if} \ 0 < x < t < 1 \\
  x \ t^2 \ 	ext{if} \ 0 < t < x < 1 
\end{cases}$.
   For $y(x) = 3 - x$, compute $K(y)$ and $K^*(y)$.  
   Ans: $K[y](x) = -\frac{x^5}{4} + \frac{4x^4}{3} - \frac{3x^3}{2} + \frac{7x}{6}$.