CHAPTER I. INTEGRAL EQUATIONS

SECTION 5: ALTERNATE 'K IS SMALL'

There is an alternate, and independent, concept of K being small which leads to convergence of the iteration process in the norm of $L^2[0,1]$. This alternate hypothesis asks that

$$\int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx < 1.$$

**THEOREM** If K satisfies the condition that

$$\int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx < 1,$$

then $\lim_p \phi_p(x)$ exists and the convergence is in norm- meaning that if $u = \lim_p \phi_p$ then

$$\lim_p \| u(x) - \phi_p(x) \| = 0.$$

**INDICATION OF PROOF.** The analysis of the nature of the convergence will go like this:

$$\| \phi_1 - \phi_0 \|^2 = \int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx \leq \int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx \leq \int_0^1 \int_0^1 f(t)^2 \, dt.$$

As before,

$$\| \phi_n - \phi_m \|^2 \leq \frac{r^{n+1}}{1-r} \| f \|^2$$

where
\begin{align*}
  r &= \int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx.
\end{align*}

**COROLLARY.** If \( r = \int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx \)
and \( u = \lim_p \phi_p \)
then
\[
|u - \phi_m|^2 \leq \frac{r^{m+1}}{1-r} |f|^2.
\]

**THE RESOLVENT.**

Before addressing the final case - where \( K \) does not have a separable kernel,
\[
\text{nor is } \int_0^1 |K(x,t)| \, dt < 1,
\]
\[
\text{nor is } \int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx < 1,
\]
we generate "resolvents" for the integral equations.

Re-examining the iteration process:
\[
\phi_0(x) = f(x),
\]
\[
\phi_1(x) = K\phi_0(x) + f(x)
\]
\[
\phi_2(x) = K(K\phi_0)x + K(f)(x) + f(x)
\]

One writes \( \phi_0 = f, \phi_1 = Kf + f, \phi_2 = K[Kf + f] + f = K^2 f + Kf + f, \) ....

In fact, with
\[
Kf(x) = \int_0^1 K(x,t) f(t) \, dt
\]
\[
K^2 f(x) = \int_0^1 K(x,t)[K(f)](t) \, dt
\]
\[
\begin{align*}
1 &= \int_0^1 K(x,t) \left[ \int_0^1 K(t,s) f(s) \, ds \right] \, dt \\
1 &= \int_0^1 \left[ \int_0^1 K(x,t) K(t,s) \, dt \right] f(s) \, ds.
\end{align*}
\]

Hence, the kernel \( K_2 \) associated with \( K^2 \) is
\[
K_2(x,t) = \int_0^1 K(x,s) K(s,t) \, ds.
\]

Inductively, \( K^n f(x) = \int_0^1 K(x,t) [K^{n-1} f](t) \, dt, \ K_n(x,t) \)
\[
= \int_0^1 K(x,s) K_{n-1}(s,t) \, ds
\]
and
\[
\phi_n(x) = f(x) + \sum_{p=1}^{\infty} K_p f(x).
\]

We have, in this section, conditions which imply that \( \sum_{p=1}^{\infty} K_p f \)
converges and that its limit \( y \) satisfies \( y = Ky + f \). Many authors call this series of operators the "resolvent" and denote
\[
R = \sum_{p=1}^{\infty} K_p.
\]

Note that \( R \) is a function which operates on elements of \( L^2[0,1] \). One writes that \( y = Ky + f \)
has solution
\[
y(x) = [/(1 + R) f](x) = f(x) + \int_0^1 R(x,t) f(t) \, dt.
\]

Suggestive algebra can be made by identifying \((1 + R)\) as
EXERCISE 15.

1. Suppose that $K(x,y) =$ \begin{cases} f(x) g(y) & \text{if } x \leq y \\ h(x) j(y) & \text{if } y \leq x \end{cases}$. Give a formula for

$$\int K(x,s) K(s,y) \, ds.$$ 

2. Compute $\int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx$ for each $K$ in the previous exercise set.

   ans: $1/12$ and $1/6$.

3. Let $K(x,t) =$ \begin{cases} 2 & \text{if } t < 1/4 \\ 0 & \text{if } t > 1/4 \end{cases}.

   For this $K$, find $y$ such that $y(x) = K[y](x) + x$. Note that

$$\int_0^1 K(x,t) \, dt < 1 \text{ and } \int_0^1 K(x,t)^2 \, dt \, dx = 1.$$ 

   What is the significance of this observation?

   ans: $x + 1/8$

4. Let $K(x,t) =$ \begin{cases} 1 & \text{if } t < x \\ 0 & \text{if } x < t \end{cases}.

   For this $K$, find $y$ such that $y(x) = K[y](x) + x$. Note that

$$\max_x \int_0^1 K(x,t) \, dt = 1 \text{ and } \int_0^1 K(x,t)^2 \, dt \, dx < 1.$$ 

   What is the significance of this observation?

   ans: $\exp(x) - 1$

5. Suppose that

$$(1 - K)^{-1} = 1 + K(1 - K)^{-1}, \text{ so that } R = K(1 - K)^{-1}.$$
so that the kernel of $K$ is $\cos(x+t)$ and the kernel of $H$ is $\sin(x+t)$. What is the kernel of $K[H]$?

ans: $\int_0^1 \cos(x+s)\sin(s+t)\,ds$.

6. Find the kernel for the resolvent of the $K$ whose kernel is $K(x,t) = xt$.
Ans: $R(x,t) = K(x,t) + K_2(x,t) + K_3(x,t) + \ldots = 3xt/2$. 