Linear Algebra, Infinite Dimensional Spaces, and MAPLE

This course will be chiefly concerned with linear operators on Hilbert Spaces. We intend to present a model, a paradigm, for how a linear transformation on an inner-product space might be constructed. This paradigm will not model all such linear mappings. To model them all would require an understanding of measures and integration beyond what a beginning science or engineering student might know.

To set the pattern for this paradigm, we first recall some linear algebra. We recall, review, and re-examine the finite dimensional setting. In that setting, there is the Jordan Canonical Form. We present this decomposition for matrices in an alternate view from the traditional one. The advantage to the representation presented here is conceptual. It sets a pattern in concepts, instead of in "form." We think of projections and eigenvalues. And, we must turn to nilpotent matrices when projections and eigenvalues are not enough.

This is how we begin.

Section 1: A Decomposition for Matrices

Definition A *projection* is a transformation P from $E \to E$ such that $P^2 = P$. Note that some texts also require that P should be non-expansive in the sense that $|Px| \le |x|$. An example of a projection that is not non-expansive is $P(x,y) = (x+y,0)$

Definition A *nilpotent transformation* is a linear transformation N from $E \to E$ for which there is an integer k such that $N^k = 0$.

Examples $\begin{pmatrix} 3 & 0 & -2 \\ -1 & 1 & 1 \\ 3 & 0 & -2 \end{pmatrix}$ is a projection from R^3 to R^3 and $\begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ is a nilpotent function from R^3 to R^3.

Theorem 1(Spectral Resolution for A) If A is a linear function from R^n to R^n then there are sequences $\{\lambda_i\}_{i=1}^{k}$, $\{P_i\}_{i=1}^{k}$, and $\{N_i\}_{i=1}^{k}$ such that each λ_i is a number and

(a) P_i is a projection, (b) N_i is nilpotent, (c) $P_iP_j = 0$ if $i \ne j$

(d) $N_iP_j = 0$ if $i \ne j$, (e) $N_iP_i = N_i$ (f) $I = \sum_{1}^{k} P_i$,

and (g) $A = \sum_{i}^{k} [\lambda_i P_i + N_i]$

<u>Outline of Proof:</u> We assume the Cayley-Hamilton Theorem which states that if A in an nxn matrix and $D(\lambda)$ is the polynomial in λ defined by $D(\lambda) = \det(\lambda I - A)$, then the polynomial in A defined by making the substitution A = λ satisfies $D(A) = 0$.

To construct the proof for the theorem, first factor $D(\lambda)$ as

$$D(\lambda) = \prod_{p=1}^{k} (\lambda-\lambda_p)^{m_p},$$

where the λp's are the zero's of D with multiplicity m_p. Now form a partial fraction decomposition of $D(\lambda)$: Let $a_1, a_2, \dots a_k$ be functions such that

$$\frac{1}{D(\lambda)} = \sum_{p=1}^{k} \frac{a_p(\lambda)}{(\lambda-\lambda_p)^{m_p}}.$$

Two Examples

$$A_1 = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$D_1(\lambda) = (\lambda-1)(\lambda-2)$ and $D_2(\lambda) = (\lambda-1)(\lambda-2)^2$

$$\frac{1}{(\lambda-1)(\lambda-2)} = \frac{-1}{\lambda-1} + \frac{1}{\lambda-2}$$

$$\frac{1}{(\lambda-1)(\lambda-2)^2} = \frac{1}{\lambda-1} + \frac{-\lambda+3}{(\lambda-2)^2}$$

If $q_p(\lambda) = \prod_{i \neq p} (\lambda-\lambda_i)^{m_i}$ **then**

$$1 = \sum_{p=1}^{k} a_p(\lambda)\, q_p(\lambda) \quad \text{so that} \quad I = \sum_{p=1}^{k} a_p(A)\, q_p(A)$$

Two Examples Continued

$1 = -1\ (\lambda-2) + (\lambda-1) = (2-\lambda) + (\lambda-1)$

$1 = (\lambda-2)^2 + (-\lambda+3)(\lambda-1)$

$$I = \begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix} + \begin{pmatrix} -2 & -2 \\ 3 & 3 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Claim 1: Using the Caley-Hamilton Theorem if $P_j = a_j(A)\, q_j(A)$ then $P_i P_j = 0$.

Claim 2: P_j is a projection since $P_j = P_j \, I = \sum_{i=1}^{k} P_j P_i = P_j^2$.

Claim 3: By the Caley-Hamilton Theorem, if

$$N_i = a_i(A)q_i(A)(A-\lambda_i I) = P_i(A-\lambda_i I)$$

then

$$N_i^{m_i} = 0.$$

Claim 4: $N_i P_i = N_i$ and $N_i P_j = 0$.

To see this note that $N_i P_j = P_j N_i = P_j P_i (A-\lambda_i I) = 0$

and $N_i P_i = P_i P_i (A-\lambda_i I) = N_i$.

Finally, since $I = \sum_i P_i$ then

$$A = \sum_i P_i A$$

$$= \sum_i P_i (\lambda_i I + A - \lambda_i I)$$

$$= \sum_i [\lambda_i P_i + N_i].$$

Two Examples Finished

$$\begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} = 1\begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix} + 2\begin{pmatrix} -2 & -2 \\ 3 & 3 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = 1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Remarks

(1) The sequence $\{\lambda_i\}_{i=1}^{k}$ is the sequence of *eigenvalues*. If x is in R^n and

$$v_i = P_i \, N_i^{m_i - 1}(x),$$

then v_i is an *eigenvector* for A in the sense that

$$\lambda_i \, v_i = A \, v_i.$$

(2) For the nilpotent part

$$m_i \le n.$$

In fact, N_i^n must be zero for each i.

Assignment **Get the spectral resolution (or Jordan Canonical Form) for the matrices:**

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -5 & 1 & 3 \\ 1 & -2 & -1 \\ -4 & 1 & 2 \end{pmatrix}$$

Section 2: Exp(tA)

Often in differential equations -- both in ordinary and partial differential equations -- one can conceive of the problem as having this form

$$Z' = AZ\ ,\ \ Z(0) = c.$$

If one can make sense of exp(tA), then surely this should be the solution to a differential equation of the form suggested. Finite dimensional linear systems beg for such an understanding. More importantly, this understanding gives direction for the analysis of stability questions in linear, and nonlinear differential equations.

Here is a review of the linear, finite dimensional case.

Theorem 2 If P is a projection, t is a number, and x is in {E, < , >}, Then the sequence

$$S(n) = \sum_{i=1}^{n} \frac{t^i P^i}{i!}(x) = \sum_{i=1}^{n} \frac{t^i}{i!}\ P(x)$$

converges in E.

Recall that whatever norm is used for R^n, if A is an nxn matrix, then there is a number B such that if x is in R^n, then $|Ax| \leq B|x|$. Moreover, the least such B is denoted $||A||$.

Definition $\exp(tA) = \sum_{i=0}^{\infty} \frac{t^i A^i}{i!}.$

Corollary 3 If P is a projection, exp(tP)(x) = (1-P)(x) + e^tP(x).

Corollary 4 Suppose that P and Q are projections, PQ = QP = 0, and if t is a number. Then

$$\exp(tP+tQ) = e^t P + e^t Q + (1-P-Q).$$

<u>**Suggestion of Proof.**</u> **With the suppositions for P and Q, P+Q is also a projection. Thus, the previous result applies.**

Observation If N is nilpotent of order m then $\exp(tN) = \sum_{i=0}^{m-1} \frac{t^i N^i}{i!}$.

Theorem 5 If A is a linear transformation from $R^n \rightarrow R^n$ and

$$A = \sum_{i}^{k} [\lambda_i P_i + N_i]$$

is as in Theorem 1, then

$$\exp(tA) = \sum_{1}^{k} \exp(\lambda_i t)P_i\left(\sum_{j=0}^{m_i-1} \frac{t^j N_i^{\,j}}{j!} \right).$$

<u>Suggestion of Proof.</u> Suppose that λ is a number. Then

$$\exp(tA) = \exp(\lambda t)\exp(t[A-\lambda I]).$$

Suppose that $A = \Sigma_i[\lambda_i P_i + N_i]$.

Then $$P_i \exp(tA) = e^{\lambda_i t} P_i \sum_{p=0}^{\infty} \frac{t^p}{p!} (A-\lambda_i I)^p$$

$$= e^{\lambda_i t} P_i \sum_{p=0}^{\infty} \frac{t^p}{p!} P_i (A-\lambda_i I)^p \text{ since } P_i = P_i^2.$$

$$= e^{\lambda_i t} P_i \sum_{p=0}^{\infty} \frac{t^p}{p!} N_i^{\,p}.$$

Thus, $\exp(tA) = 1\ \exp(tA)$

$$= \Sigma_i P_i \exp(tA)$$

$$= \Sigma_i e^{\lambda_i t} P_i \sum_{p=0}^{m_i-1} \frac{t^p}{p!} N_i^{\,p}.$$

Assignment

Solve $Z' = AZ$, $Z(0) = c$, where A is any of the matrices in the previous assignments.