Section 11: Applications to Differential and Integral Equations

This section will place some of the ideas that have come before into the context of integral equations and into the context of ordinary differential equations with boundary conditions. We have already seen that if A is a bounded, linear transformation on E then exp(tA)(c) provides the solution for the initial value problem

$$Y' = AY, \quad Y(0) = c.$$

Example

Let $$K(x,y) = \begin{cases} x(1-y) \text{ if } 0 \le x \le y \le 1 \\ y(1-x) \text{ if } 0 \le y \le x \le 1 \end{cases}$$

and let $$A(f)(x) = \int_0^1 K(x,y)\, f(y)\, dy.$$

(1) A is a bounded linear transformation from $L^2[0,1]$ to $L^2[0,1]$ that is self-adjoint.

(2) If f is in $L^2[0,1]$ then these are equivalent:

(a) $g(x) = \int_0^1 K(x,y)\, f(y)\, dy$, **so that** $g = A(f)$,

and

(b) $g'' = -f$ **with** $g(0) = g(1) = 0$.

Define B to have domain the functions g in $L^2[0,1]$ with $g(0) = g(1) = 0$ and with two derivatives and $B(g) = -g''$. Note that $A(f) = g$ if and only if $f = B(g)$.

(3) These are equivalent:

(a) λ is a number, φ is a function, and $\lambda\varphi = A(\varphi)$, and

(b) $\lambda = \frac{1}{n^2\pi^2}$ **and** $\varphi(x) = \sqrt{2}\,\sin(n\pi x)$, **for some positive integer n.**

(4) If f is in $L^2[0,1]$, then $A(f) = \sum_1^\infty \frac{1}{n^2\pi^2} \langle f, \varphi_n\rangle \varphi_n$ **where φ_n is as above.**

(5) (a) $B(g) = \sum_{p=1}^\infty p^2\pi^2 \langle g, \varphi_p\rangle \varphi_p$.

(b) The solution for $Z'(t) + BZ(t) = 0$, $Z(0) = C$ is

$$\sum_{1}^{\infty} \exp(-n^2 \pi^2 t) \langle C, \varphi_n \rangle \varphi_n.$$

(6) The system $Y' = AY + F$, $E_0Y(0) + E_1Y(1) = 0$ can be rewritten as an initial value problem provided $[E_0 + E_1 \exp(A)]$ has an inverse. In any case, a *Green's function* G can be constructed where by, for appropriate functions F, solutions for the system can be computed as

$$Y(x) = \int_0^1 G(x,t)\, F(t)\, dt.$$

Assignment Find matrices A, E_0 and E_1 such that the boundary value problem

$$y'' + 3y' + 2y = f,$$
$$y(0) + y(1) = 0,$$
$$y'(0) - y'(1) = 0$$

is written as a system $Y' = AY + F$, $E_0Y(0) + E_1Y(1) = 0$.

Example: Here are graphs of the solution y for

$$y'' + 3y' + 2y = f,$$

with $y(0) = y'(0) = 0$, and another solution with $y(0) = 0$, $y(1) = 0$, and another solution with $y(0) - y(1) = 0$, $y'(0) + y'(1) = 0$.

Is it easy to identify which graph goes with which solution?

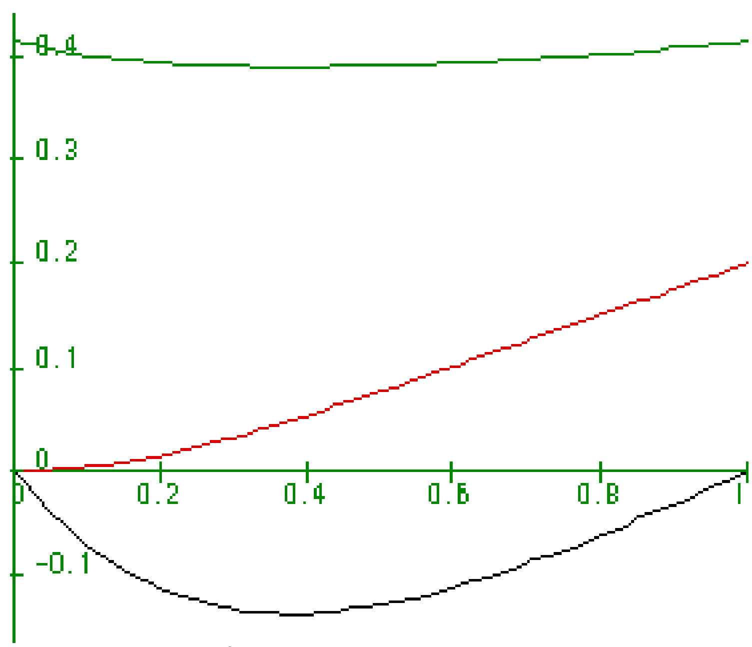

Figure 11.1

MAPLE Remark: **It is interesting to think of how the graphs of the solutions to**

$$y'' + 3\,y' + 2\,y = 1$$

on [0,1] change as the boundary conditions change. At each time the relationship specified in the differential equation between y and its derivatives must hold. How y starts at t = 0 is modified in order to satisfy the boundary conditions. Here are three examples.

```
> deq:=(D@@2)(y)(t) + 3*D(y)(t) + 2*y(t) = 1;
> bdry1:=y(0) = 0, D(y)(0)=0;
  bdry2:=y(0) = 0, y(1) = 0;
  bdry3:=y(0) = y(1), D(y)(0) = -D(y)(1);
> y1:=dsolve({deq,bdry1},y(t));
  y2:=dsolve({deq,bdry2},y(t));
  y3:=dsolve({deq,bdry3},y(t));
```

It seems that MAPLE does not know how to solve this last boundary value problem with boundary conditions bdry3. Not to worry. Humans can teach MAPLE!

```
> solve({ a+b = a*exp(-1)+b*exp(-2),
                  -a-2*b = -(-a*exp(-1)-2*b*exp(-2))},{a,b});
> assign(");
> y3:=t->1/2+a*exp(-t) +b*exp(-2*t);
```

Here's my check that y3 is a solution:

```
> (D@@2)(y3)(t) + 3*D(y3)(t) + 2*y3(t);
> evalf(y3(0)-y3(1));
> evalf(D(y3)(0)+D(y3)(1));
```

Finally, here is a plot of the three solutions.

```
> plot({rhs(y1),rhs(y2),y3},0..1);
```

Section 12: The Simple Paradigm for Linear Mappings from E to E

It's time to put together two ideas: might one dare to guess the form for the paradigm in a general inner product space? And, what is special about the paradigm in case the linear function it represents is a *bounded* linear mapping? The answer to both questions is as could be expected.

Example Suppose that $\{\varphi_p\}_{p=1}^{\infty}$ is an infinite maximal orthonormal sequence and that $\{\lambda_p\}_{p=1}^{\infty}$ is a sequence of numbers. The linear transformation A given by

$$Ax = \sum_{p=1}^{\infty} \lambda_p \langle x, \varphi_p \rangle \varphi_p$$

is bounded if $\{\lambda_p\}_{p=1}^{\infty}$ is bounded and unbounded, otherwise. To verify this, we note that $|Ax| \leq \sup_p(|\lambda_p|)\ |x|$ for all x in E and that $|A(\varphi_p)| = |\lambda_p|\ |\varphi_p|$. The domain of A is all x for which

$$\sum_{p=1}^{\infty} |\lambda_p \langle x, \varphi_p \rangle|^2 < \infty.$$

One can show that if $\{\lambda_p\}_{p=1}^{\infty}$ is bounded then A has domain all of E and that the domain of A is dense in E in any case. It should be noted that the domain of A^2 is a (perhaps proper) subset of the domain of A. (See the Remark below.)

Moreover, these are equivalent:
(a) A has an inverse, and
(b) each $\lambda_p \neq 0$.

Finally, if

$$Bx = \sum_{p=1}^{\infty} \mu_p \langle x, \varphi_p \rangle \varphi_p,$$

then A and B commute.

Remark To see that the domain of A^2 is a subset of the domain of A, let x be in the domain of A^2. Define a sequence $\{\mu_p\}$ by

$$\mu_p = \lambda_p \text{ if } |\lambda_p| \geq 1 \text{ and } = 0 \text{ if } |\lambda_p| < 1.$$

Suppose x is in $D(A^2)$. Then

$$\infty > \Sigma_p |\lambda_p^2 <x,\varphi_p>|^2 \geq \Sigma_p |\mu_p^2 <x,\varphi_p>|^2$$
$$\geq \Sigma_p |\mu_p <x,\varphi_p>|^2 .$$

Then, $||x||^2 + ||A^2x||^2 = \Sigma_p |<x,\varphi_p>|^2 + \Sigma_p |\lambda_p^2 <x,\varphi_p>|^2$
$\geq \Sigma_p |<x,\varphi_p>|^2 + \Sigma_p |\mu_p <x, \varphi_p>|^2 \geq \Sigma_p |\lambda_p <x, \varphi_p>|^2 = |A(x)|^2.$

Assignment
(12.1) Give an example of an unbounded linear function whose inverse is bounded.

(12.2) Give an example of an unbounded linear function whose inverse is unbounded.

(12.3) Give the simple paradigm representation for $A:D \quad L^2[0,1] \to L^2[0,1]$ **where** $D = \{f: f'' \text{ exists and } f(0) = f(1) = 0\}$ **and** $A(f) = f''$.

MAPLE remark. We use MAPLE to verify that the sequence $f_n(x) = \sin(n\pi x)$ **forms a set of eigenfunctions for A as defined in Assignment 12.3. We see that** $f_n(0) = 0$ **and** $f_n(1) = 0$.

```
> diff(sin(n*Pi*x),x,x);
```

The function A can be represented in this simple paradigm as we saw in Assignment 12.3. With MAPLE's help do an experiment concerning A.

Consider the linear mapping

$$A(f) = \sum_{n=1}^{\infty} -2\, n^2 \pi^2 < f(t), \sin(n\pi t) > \sin(n\pi x).$$

We examine $A(\, t\ (t-1)\,)$.

```
> AProx:=x->2*sum(-n^2*Pi^2*int(t*(t-1)*sin(n*Pi*t),t=0..1)*sin(n*Pi*x),
n=1..10);
> plot(AProx(x),x=0..1);
```

This last was an approximation for taking two derivatives of $t\ (t-1)$**. Surely, it is irresistable to take the inverse of A, or at least to approximate this inverse.**

```
> APverse:=x->2*sum(-int(t*(t-1)*sin(n*Pi*t),t=0..1)/
                              (n^2*Pi^2)*sin(n*Pi*x), n=1..10);
> plot({x^4/12-x^3/6+x/12+.0001,APverse(x)},x=0..1);
```

The question is, why did APverse turn out to be

$$g(x) \approx x^4/12 - x^3/6 + x/12$$

instead of, say $x^4/12 - x^3/6$**? Two derivatives of both are zero....**