Section 13: Adjoint Operators

Not every bounded linear mapping is self adjoint. You know this from your experience with matrices. Never mind. They all have adjoints!

Suppose that A is a bounded linear transformation on E. Fix z in E and define L by L(x) = <Ax, z> for all x in E. L is a bounded linear function from E to $\mathbb{C}$ By the Riesz Theorem, there is y in E such that L(x) = <x,y> for all x in E. Consider this pairing of z and y: for each z there is only one y. We define the adjoint of A by pairing this y with z.

Definition If A is a bounded linear function on E then the *adjoint of A*, A*, is the function on E such that <Ax, z > = <x, A*z> for all x in the domain of A.

Observations
(1) A* is a function from E to E that is linear and bounded.
(2) N(A*) = R(A)

(3) If

$$Ax = \sum_{p=1}^{\infty} \lambda_p \langle x, \varphi_p \rangle \varphi_p$$

for all x then

$$A^*(y) = \sum_{p=1}^{\infty} \lambda_p^* \langle y, \varphi_p \rangle \varphi_p .$$

Theorem 26 Suppose that K is a continuous function on [0,1]x[0,1] to R, so that K(x,y) = K(y,x)*. Suppose also that A is defined by

$$A(f)(x) = \int_0^1 K(x,y)f(y)\, dy.$$

Then A is self adjoint. Moreover, if M = lub{<Ax, x>: |x| = 1} and m = glb{<Ax ,x>: |x| = 1} and λ is an eigenvalue for A then $m \le \lambda \le M$.

Assignment
(13.1) Suppose that K is a continuous function from [0,1]x[0,1] to $\mathbb{C}$ If

$$A(f)(x) = \int_0^1 K(x,y)\, f(y)\, dy$$

then A is a bounded, linear function from all of $L^2[0,1]$. Moreover, A* is given by

$$A^*(g)(u) = \int_0^1 H(u,v)\, g(v)\, dv,$$

where $H(u,v) = K(v,u)^*$, so that A is self-adjoint if $K(x,y) = K(y,x)^*$.

(13.2) Let

$$K(x,y) = \begin{cases} \frac{1}{2}(y-x) - \frac{1}{4} & \text{if } x < y \\ \frac{1}{2}(x-y) - \frac{1}{4} & \text{if } y < x \end{cases}.$$

Define

$$A(f)(x) = \int_0^1 K(x,y)\, f(y)\, dy$$

(a) Show that A is a bounded, linear function from $L^2[0,1]$ to $L^2[0,1]$.
(b) Show that $A^* = A$.
(c) Show that these are equivalent:
 (a) $g = A(f)$
 (b) $g'' = f$, $g(0) + g(1) = 0$, $g'(0) + g'(1) = 0$.
(d) Find the eigenvalues and eigenfunctions for A.
ans: $\lambda_n = - 1/[(2n+1)\pi]^2$, $f_n(x) = \cos((2n+1)\pi x)$ and $\sin((2n+1)\pi x)$

(e) Show that A has the property that if $\{\lambda_p\}_{p=1}^{\infty}$ is the sequence of eigenvalues then $\lim_p \lambda_p = 0$.

MAPLE remark: Likely, by now, the reader is aware that what analysts call the adjoint of an operator is different from what one often sees in texts on linear algebra that is called the adjoint of a matrix. We persist. But, we take note that the linear algebra package in MAPLE follows the notation of the linear algebra texts, just as we follow the precedence of Hilbert space texts. Perhaps this MAPLE exercise will give understanding to the two "adjoints".

```
> with(linalg):
> A:=array([[1,2],[3,4]]);
> Ajoint:=adjoint(A); Apose:=transpose(A);
> evalm(A &* Ajoint)/det(A);
> dotprod(evalm(A &* vector([a,b])),vector([x,y])) -
                dotprod(vector([a,b]),evalm(Apose &* vector([x,y])));
> simplify(");
```

Thus, the transpose is the adjoint of these notes. Here is a complex example.

```
> A:=array([[1+I,2+3*I],[3-2*I,4]]);
> Ajoint:=adjoint(A);
  Apose:=transpose(A);
  Aconjpose:=transpose(map(evalc,map(conjugate,A)));
> evalm(A &* Ajoint)/det(A);
> dotprod(evalm(A &* vector([a,b])),vector([x,y]))
            - dotprod(vector([a,b]), evalm(Aconjpose &* vector([x,y])));
> simplify(");
```

Section 14: Compact Sets

During the next portion of the notes, we will begin an investigation of linear, compact operators. In order to do this study, there are several ideas about number sets that should be remembered: what is a *bounded* number set, what is a *compact* number set, what is a *sequentially compact* number set, and what is a *totally bounded* number set. These notions carry over to normed spaces, too

Definitions A set S is *bounded* if there is a number b such that if x is in S then $|x| \leq b$.

A set C is *compact* if every open covering of C has a finite subcovering.

A set C is *sequentially compact* provided that if $\{s_p\}$ is a sequence with values in C then there is a subsequence of $\{s_p\}$ that converges and has limit in C.

A set S is *totally bounded* if, for each positive number c, there is a finite set of points $\{x_p\}_{p=1}^{n}$ such that S is contained in ${}_p\, D_c(x_p)$. (Here, and in the remainder of the notes, $D_c(s)$ represents the open disk with center s and radius c.)

Examples

(1) Unbounded sets are not totally bounded.

(2) In L^2, $cl(D_1(0))$ - - the closed unit disk - - is not totally bounded. Here's why: Let $c = \sqrt{2}/2$. Since $\|e_i - e_j\| = \sqrt{2}$, then any collection of disks of radius less than $\sqrt{2}/2$ which covers all of $cl(D_1(0))$ must be infinite.

(3) In R^3, $cl(D_1(0))$ is totally bounded. Here's why: Suppose that K is a positive integer. Choose points F such that if {a,b,c} is one of them then each of a, b, and c has the form m/k where m is an integer and $-k \leq m \leq k$. For example, if k = 3, then {a,b,c} might be {2/3, 1/3, -2/3}. If {x,y,z} is any point in $cl(D_1(0))$, then there is a point {a,b,c} in F such that

$$\|\{x,y,z\} - \{a,b,c\}\|^2 < \frac{3}{k^2}.$$

Thus, if c > 0, choose k such that

$$\sqrt{\frac{3}{k^2}} < c \text{ or } \frac{\sqrt{3}}{c} < k.$$

We cover $cl(D_1(0))$ by disks with radius c and centers at the points of F.

(4) (0,1) and $[0,\infty)$ are not sequentially compact.

Remark: In finite dimensions, if S is a set, then these are equivalent:
(a) S is compact,
(b) S is sequentially compact, and
(c) S is closed and bounded.

Our use of these ideas is illustrated in the next Theorem 27. To show the generality, we reference *Introductory Real Analysis*, by Kolmogorov and Fomin (translated by Richard Silverman) and published by Dover. In a section numbered 11.2, they prove a Theorem 2: A metric space is compact if and only if it is totally bounded and complete.

Theorem 27 If S is a subset of the Hilbert space {E, < , >}, then these are equivalent:
(a) S is sequentially compact, and
(b) S is closed and totally bounded.

<u>Suggestion</u> <u>of</u> <u>Proof:</u>

a   b Prove S is closed. Suppose lim u_p = v, each u_p is in S. Sequential compactness implies that v is in S.

To prove that S is totally bounded: Suppose $\varepsilon > 0$. Pick u_1; if S is not contained in $D_\varepsilon(u_1)$ pick u_2 in S but not in $D_\varepsilon(u_1)$. If there are an infinite number of such disks then continue this process. This produces an infinite sequence $\{u_p\}$ such that $|u_p - u_q| > \varepsilon$. S being sequentially compact implies that this sequence has a subsequence that converges. This is a contradiction and there must be only a finite number of such disks.

b   a Suppose that S is totally bounded and that $\{u_p\}$ is an infinite sequence with values in S. A convergent subsequence will be extracted. Choose a finite number of disks with radius 1 that cover S. An infinite subsequence of u lies in one of these. Call it $u_1(p)$. Cover this disk containing u_1 with a finite number of disks with radius 1/2. An infinite subsequence of u_1 lies in one of them, call this $u_2(p)$. ETC. Then $|u_n(n) - u_m(m)| < 1/n$ for all m > n. This is a subsequence of u that converges. Since the space is complete, the sequence has a limit. If the set is closed, the limit is in S.

Remark

Insight into the geometric structure of a sequentially compact set in L^2 is gained by realizing that, while a sequentially compact set in L^2 may be infinite dimensional, it can contain no open set.

Assignment

(1) List four points {a,b} such that if 0 < x < 1, 0 < y < 1, then

$$\| \{x,y\} - \{a,b\} \| < \frac{1}{2}$$

for at least one of the four {a,b}.

(2) Show that the Hilbert cube is totally bounded. The Hilbert cube consist of points $\{x_p\}$ in L^2 such that $|x_n| \le \frac{1}{2^n}$.

(Hint: if c > 0, $\frac{1}{2^n} < c/2$, and x is in the cube, then there is x_n and r_n such that $x = x_n + r_n$, $|r_n| < c/2$, and x_n is in $D_1(0) \cap R^n$. This last set is totally bounded.)

MAPLE remark: The compact sets in R^n are precisely the closed and bounded sets. That is, a set in R^n is compact if and only if it is closed and bounded. In a Hilbert space, every sequentially compact set is closed and bounded, but there are closed and bounded sets that are not sequentially compact. In fact, the closed unit disk is such an example. The following syntax verifies that the sequence

$$\sqrt{2}\,\sin(n\pi x),\quad n = 1, 2, 3, \ldots$$

in an infinite sequence in $L^2[0,1]$ in the unit disk, any two are orthogonal, and the distance between any two is the square-root of two.

```
> int(2*sin(n*Pi*x)^2,x=0..1);
> int(2*sin(n*Pi*x)*sin(m*Pi*x),x=0..1);
> int(2*(sin(n*Pi*x)-sin(m*Pi*x))^2,x=0..1);
> simplify(");
> sqrt(simplify((4*n^3*m-4*n*m^3)/(n*m*(n^2-m^2))));
```

Note the relationship between this example and Example 2 of this section.