Section 15: Compact Operators

If the simple paradigm for a linear operator in a Hilbert space is to be useful, we should be able to look for characteristics of the paradigm that determine characteristics of the operators. We now ask what conditions on the simple paradigm will imply that the image of the unit disk is contained in a sequentially compact set.

Theorem 28 Suppose that $\{\varphi_p\}_{p=1}^{\infty}$ is a maximal orthonormal sequence in E, $\{\lambda_p\}_{p=1}^{\infty}$ is a sequence of numbers, and

$$Ax = \sum \lambda_p <x, \varphi_p> \varphi_p.$$

These are equivalent:
(1) $\lim_p |\lambda_p| = 0$, and
(2) $cl(A(D_1(0)))$ is sequentially compact.

<u>Suggestion of Proof.</u>
(1) (2). We show the closure of $A(D_1(0))$ is totally bounded. Since $|\lambda_p| \to 0$ there is a number B such that $|\lambda_p| < B$ for all p. Then $A(D_1(0))$ is contained in the closed ball with radius B for

$$|A(x)|^2 = \sum |\lambda_p|^2 |<x, \varphi_p>|^2 \leq B^2 |x|^2.$$

We now pick out the set of points required by the definition of totally bounded: For $1 > c > 0$, choose N such that $\max_{N \leq p} |\lambda_p| < c/2$. Since the closed disk with radius B is a sequentially compact subset of R^N choose a finite set of points $\{c_p\}$ in R^N such that if α is in R^N with $|\alpha| \leq B$ then $|\alpha - c_p|^2 < c/2$ for some p. Now pick y in the closure of $A(D_1(0))$. There is x in $D_1(0)$ such that $Ax = y$. There is p such that

$$\left| \sum_{q=1}^{N} [<y,\varphi_q> - c_{pq}]^2 \right| < c/2.$$

Also,

$$|y-c_p| = \left| \sum_{q=1}^{N} [<y,\varphi_q> - c_{pq}]^2 \right| + \sum_{N+1}^{\infty} |<y,\varphi_q>|^2$$

$$\leq c/2 + \sum_{N+1}^{\infty} |\lambda_q|^2 |<x,\varphi_q>|^2 \leq c/2 + c|x|^2/2 = c.$$

(2) (1). Supppose $\lim |\lambda_p| \neq 0$. There is an infinite sequence $\{\varphi_n\}$ with $|\varphi_n| = 1$ and $|\lambda_n \varphi_n| \geq c$. Let x_n be this sequence φ_n. Then

$|A(\varphi_n) - A(\varphi_m)|^2 = |\lambda_n|^2 + |\lambda_m|^2 \geq 2c^2$. **Thus, $A(x_n) = A(\varphi_n)$ has no convergent subsequence.**

DEFINITION The linear operator A is *compact* if $A(D_1(0))$ is contained in a closed and totally bounded set.

Remarks.

(1) Note that the statement that the linear operator is compact is equivalent to the following:
The linear operator is compact provided if $\{x_p\}$ is a bounded sequence in the domain of A, then there is a subsequence $\{x[u_p]\}$ such that $\{A(x[u_p])\}$ converges.

(2) If $\{x_p\}$ is bounded and converges weakly to y and A is a compact linear operator, then $\{A(x_p)\}$ converges strongly to A(y).

(3) Every compact linear operator is continuous.

Assignment

(15.1) Let $\{\varphi_p\}_{p=1}^{\infty}$ be a maximal orthonormal sequence and define A from E to E by

$$Ax = \sum (2 - \frac{1}{p}) < x, \varphi_p > \varphi_p.$$

(a) Show that A is self-adjoint, bounded, linear, and not compact.
(b) Show that there is no z such that $Az = 2z$.
(c) Show that $||Ax|| \leq 2||x||$ for x in E and if $b < 2$ then there is y in E such that $||Ay|| > b\ ||y||$.
(d) Solve the differential equation $Z' = AZ$, $Z(0) = C$ E.

(15.2) Define B from E to E by

$$Bx = \sum p < x, \varphi_p > \varphi_p.$$

(a) Show that B is self adjoint, unbounded, linear, and not compact.
(b) Show that there is z such that $Bz = 2z$.
(c) Give a point x not in the domain of B.
(d) Let z be the solution of the differential equation $Z' + BZ = 0$, $Z(0) = C$ E. Show that the solution is contractive in the sense that $|Z(t)| \leq |C|$.

(15.3) Solve these two equations

(a) $\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}$, $z(t,0) = z(t,1) = 0$, $z(0,x) = f(x)$ $L^2[0,1]$

(b) $\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}$ **+ g(t,x), z(t,0) = z(t,1) = 0, z(0,x) = f(x), with f and g(t,) in $L^2[0,1]$.**

MAPLE remark: We provide a solution for the partial differential equation

$$\partial z/\partial t = \partial^2 z/\partial x^2,$$
$$z(t,0) = z(t,1) = 0 \text{ for all } t > 0,$$
$$z(0,x) = x\ (x-1) \text{ for } 0 < x < 1.$$

The methods we will use employ the techniques of this course. This is an appropriate place to embed an understanding of the methods of solving partial differential equations by separation of variables into the context of these notes.

First, we write this partial differential equation as an ordinary differential equation in $L^2[0,1]$. There Z' = AZ where Af = f'' with f(0) = f(1) = 0. We have seen the representation for A in Assignment 12.3:

$$A[f](x) = \sum_{n=1}^{\infty} -2\, n^2 \pi^2 < f(s), \sin(n\pi s) > \sin(n\pi x).$$

Maple computes the solution and draws the surface of the solution.

```
> sum(2*int(s*(s-1)*sin(n*Pi*s),s=0..1) *sin(n*Pi*x),n=1..11);
> z:=(t,x)->sum(exp(-n^2*Pi^2*t)*(2*int(s*(s-1)*sin(n*Pi*s),s=0..1))
                              *sin(n*Pi*x),n=1..11);
> plot3d(z(t,x),t=0..1/2,x=0..1,axes = NORMAL,orientation=[150,65]);
```

Section 16: The Space of Bounded Linear Transformations

One goal has been, and continues to be, the investigation of eigenvalue problems. That is, we wish to establish whether we can find a vector v and a number λ such that

$$Av = \lambda\, v.$$

We must do this if we are to generate a paradigm representation for linear transformation A.

We will see that if A is self-adjoint and compact, then there are solutions for an eigenvalue problem. En route, we need to consider the *operator topology*. The notion of a norm for a matrix has been suggested in these notes already.

Definition BLT(E_1,E_2) is the space of *bounded, linear transformations from E_1 to E_2*. If L is in the space then $||L||_{1,2}$ is the smallest number b such that $|L(x)|_2 \le b\ |x|_1$ for all x in E_1.

Example

(1) Suppose that $\{\varphi_p\}_{p=1}^{\infty}$ is a maximal orthonormal sequence in E_1 and that $\{\psi_p\}_{p=1}^{\infty}$ is a maximal orthonormal sequence in E_2. Define

$$Lx = \sum (2 - \frac{1}{p}) \langle x, \varphi_p\rangle\, \psi_p.$$

It follows that $||L||_{1,2} = 2$.

(2) Show that if $\lim_p \alpha_p = 0$ then $\lim_n \sum_{p=1}^{n} \alpha_p \langle\ \ , \varphi_p\rangle\, \varphi_p$ converges in BLT and otherwise, $\lim_n \sum_{p=1}^{n} \alpha_p \langle x, \varphi_p\rangle\, \varphi_p$ strongly -- in the sense of Section 5.

(3) Let $\{\alpha_p\}_{p=1}^{\infty}$ be a point in L^2 and $\{\varphi_p\}_{p=1}^{\infty}$ be a maximal orthonormal sequence in $L^2([0,1])$. Define $K_n(x,y) = \sum_{p=1}^{n} \alpha_p\, \varphi_p(x)\, \varphi_p(y)$ on [0,1]x[0,1]. Then

$$\int_0^1\int_0^1 |K_m(x,y) - K_n(x,y)|^2\, dx\, dy = \sum_{n+1}^{m} |\alpha_p|^2.$$

Hence $\{K_p\}_{p=1}^{\infty}$ is Cauchy in $L^2([0,1]x[0,1])$ and has limit

$$K(x,y) = \sum_{p=1}^{\infty} \alpha_p \varphi_p(x)\ \varphi_p(y).$$

Define K_n from $L^2[0,1]$ to $L^2[0,1]$ by

$$K_n(f)(x) = \int_0^1 K_n(x,y)\ f(y)\ dy.$$

Then

$$\int_0^1 \left(\int_0^1 [K_m(x,y) - K_n(x,y)]\ f(y)\ dy \right)^2 dx$$

$$\leq \int_0^1 \left(\int_0^1 [K_m(x,y) - K_n(x,y)]^2\ dy \right) \int_0^1 |f(y)|^2\ dy\ \ dx$$

$$\leq ||\ K_m - K_n\ ||^2\ ||f||.$$

Therefore, $\{K_p\}_{p=1}^{\infty}$ is Cauchy in BLT on $L^2[0,1]$.

Theorem 29 Suppose that $\{L_p\}_{p=1}^{\infty}$ is a sequence of compact operators in BLT and converges to K in the norm of BLT. Then K is compact.

Suggestion of Proof: First, note that K is a *bounded* linear operator because corresponding to 1 > 0, there is N such that if $n \geq N$ then

$$|L_n(x) - K(x)| \leq 1\ \ |x|,$$

so that $\quad |K(x)| \leq |L_n(x)| + |K(x) - L_n(x)| \leq |L_n(x)| + 1\ \ |x|.$

We now need to show that the closure of $K(D_1(0))$ is totally bounded. Let $\varepsilon > 0$. We seek a finite number of ε disks that will cover the closure of $K(D_1(0))$. Let N be such that if $n > N$ then $|L_n(x) - K(x)| < \varepsilon\ |x|/2$. Since L_n is compact there is a finite sequence $\{c_p\}$ such that

$${}_pD_{\varepsilon/2}(c_p) \quad Cl(L_n(D_1(0))).$$

Let x $\quad D_1(0)$. Then

$$|K(x) - c_p| \leq |K(x) - L_n(x)| + |L_n(x) - c_p|.$$

Hence, $Cl(\ K(D_1(0))) \quad {}_pD_{\varepsilon/2}(c_p)$.

Remark: We showed that if $\lim_{p\to\infty}|\lambda_p| = 0$ and $\{\varphi_p\}$ is a maximal orthonormal sequence then

$$L_n = \sum_{p=1}^{n} \lambda_p < \ ,\varphi_p> \varphi_p$$

converges in the BLT norm to

$$\sum_{p=1}^{\infty} \lambda_p < \ ,\varphi_p> \varphi_p.$$

Moreover, each L_n is a compact operator. This provides another example of a sequence of compact operators that converges in the BLT norm and the limit is compact.

Example Let $K:[0,1]x[0,1] \to \mathbb{C}$ be continuous and let

$$K(f)(x) = \int_0^1 K(x,y)\, f(y)\, dy.$$

Then K is compact.

Theorem 30 If A is a bounded, linear, self adjoint operator, then these are equivalent:

(1) $b > 0$ and $|Ax| \le b\ |x|$ for all x in E, and

(2) $b \ge \sup\{\ |<Ax,x>|\ :\ |x| = 1\}$

Proof: (1)   (2) This follows from the Cauchy-Schwartz inequality.

(2)   (1) Let α be the number $\sqrt{|Ax|/|x|}$ and z be the vector Ax/α.

$|Ax|^2 = <Ax,Ax> = < \alpha\ A(x), z >$

$= 1/4\ [\ <A(\alpha x+z), \alpha x+z> - <A(\alpha x-z), \alpha x-z>\ ]$

$\le 1/4\ b\ (\ |\alpha x+z|^2 + |\alpha x-z|^2\ )$ and expanding these dot-products

$= 1/2\ b\ (\ |\alpha x|^2 + |z|^2)$

$= 1/2\ b[\ \alpha^2 |x|^2 + 1/\alpha^2\ |Ax|^2]$ and using the definition of α

$= b\ |x|\ |Ax|$.

Thus, $|Ax| \le b|x|$.

Example: A part of the hypothesis for Theorem 30 is that A is self adjoint. This part of the hypothesis was not used in getting that (1) implies (2). However, that part cannot be completely removed, as is illustrated by the following example. Suppose that φ_1 and φ_2 are orthonormal. Take

$$Ax = < x, \varphi_1 > \varphi_2.$$

Then A is linear, and

$$|Ax| = |< x, \varphi_1 >| \le |x|.$$

However, $\quad |< Ax, x >|^2 = |< x , \varphi_1> < x , \varphi_2>|^2$

and this is strictly less than

$$|< x , \varphi_1>|^2 + |< x , \varphi_2>|^2 \le |x|^2.$$

For example, if $A(x,y) = (y,0)$, then A is bounded, not self adjoint and

$$<A(\{x,y\}), \{x,y\}> = x\ y.$$

The maximum value of x y subject to $x^2 + y^2 = 1$ is 1/2, not 1 -- which is ||A||.

Examples: Mindful that some authors take all linear projections to be bounded (and orthogonal) here are two examples of linear projections that are *unbounded*: The second one has range a finite dimensioned space.

Example (1) (Due to Mary Chamlee, Fall,'92) E = L^2 and
$P(\{x_1, x_2, x_3, \ldots , x_n, \ldots\}) = \{x_1 - x_2, 0, x_3 - 2\, x_4, 0, x_5 - 3\, x_6, 0, \ldots \}$.

Example (2) (Due to Mahdi Zaidan, Fall, '92) E = L^2 and
$P(\{x_1, x_2, x_3, \ldots , x_n, \ldots\}) = \{x_1 + 2x_3 + 3x_4 + \ldots , 0, 0, \ldots \}$.

Assignment:
Suppose that K is defined by

$$K(f)(x) = \int_0^1 \cos(\pi(x-y))\, f(y)\, dy.$$

Show that K has finite dimensional range. Argue that it is a compact operator. Compute ||K||.

MAPLE remarks. We have already suggested that MAPLE can find matrix norms. In the next section, the concern will be in getting eigenvalues for self-adjoint linear mappings. A hint for how this will go is suggested in these MAPLE experiments.

```
> with(linalg):
> A:=array([[2,1],[1,2]]);
> evalf(norm(A,1));evalf(norm(A,2));evalf(norm(A,infinity));
> eigenvals(A);
> A:=array([[2,1,0],[1,0,0],[0,0,5]]);
> evalf(norm(A,1));evalf(norm(A,2));evalf(norm(A,infinity));
> lambda:=eigenvals(A);
```