Section 17: The Eigenvalue Problem

We now have the proper space in which to solve the eigenvalue problem: the space of the bounded linear transformations. We proceed to show that the eigenvalue problem can be solved.

Theorem 31 Suppose that A is a compact, self-adjoint operator. There is a real number λ and a non-zero vector z such that $|\lambda| = ||A||$ and $Az = \lambda z$.

Proof: By Theorem 30, there is a sequence $\{x_p\}$ in E such that $|x_p| = 1$ and $\lim_p \langle Ax_p, x_p\rangle = \lambda$ where $|\lambda| = ||A||$. Since A is compact, there is z in E and a subsequence of x such that $\lim_p Ax_{u(p)} = z$. Then

$$\lim_p \langle Ax_{u(p)}, x_{u(p)}\rangle = \lambda.$$

$$0 \le |Ax_n - \lambda x_n|^2 = |Ax_n|^2 + |\lambda x_n|^2 - \lambda \langle Ax_n, x_n\rangle - \lambda \langle x_n, Ax_n\rangle$$
$$\le (||A||^2 + |\lambda|^2) - 2\lambda \langle Ax_n, x_n\rangle = 2\lambda^2 - 2\lambda \langle Ax_n, x_n\rangle \to 0.$$

Thus, $\lambda \lim_p x_{u(p)} = \lim_p Ax_{u(p)} = z,$

or, $0 = A(z - \lambda \lim_p x_{u(p)}) = Az - \lambda A(\lim_p x_{u(p)}) = Az - \lambda z.$

Theorem 32 If A is a compact, self adjoint operator and $\alpha > 0$, then the number of eigenvalues λ with $|\lambda| > \alpha$ is finite. Moreover, if λ is an eigenvalue for A and $M = \{x: Ax = \lambda x\}$ then M is finite dimensional.

<u>Proof:</u> $|Ax_p - Ax_q|^2 = |\lambda_p x_p - \lambda_q x_q|^2 = |\lambda_p|^2 |x_p|^2 - |\lambda_q|^2 |x_q|^2 > \alpha$ would lead to a contradiction if there are infinitely many and A is compact.

Corollary 33 If A is a compact, self-adjoint operator, then there is a sequence $\{\varphi_p\}$ of orthonormal vectors and a number sequence $\{\lambda_p\}$ such that

$$A(x) = \sum_{p=1}^{\infty} \lambda_p \langle x, \varphi_p\rangle \varphi_p.$$

<u>Proof:</u> We have established that A has an eigenvalue λ with $|\lambda| = ||A||$. Let $M_1 = \{x: Ax = \lambda x\}$. We also know that M_1 is finite dimensional. Let $\{\varphi_p\}_1^N$ be an orthonormal basis for M_1. Then $A(\varphi_p) = \lambda \varphi_p$ for each p. Let $\{\theta_p\}_{N+1}^{\infty}$ be the additional orthonormal vectors to make a maximal orthonormal family. Let N be the (possibly infinite) combinations of $\{\theta_p\}_{N+1}^{\infty}$. Then A maps N into N. To see this, it suffices to see that $A(\theta_i)$ is in N for each i. Suppose not. Then

$$A(\theta_i) = \sum_{1}^{N} \langle A(\theta_i), \varphi_p\rangle \varphi_p = \sum_{1}^{N} \langle \theta_i, A\varphi_p\rangle \varphi_p = \sum_{1}^{N} \lambda \langle \theta_i, \varphi_p\rangle \varphi_p = 0.$$

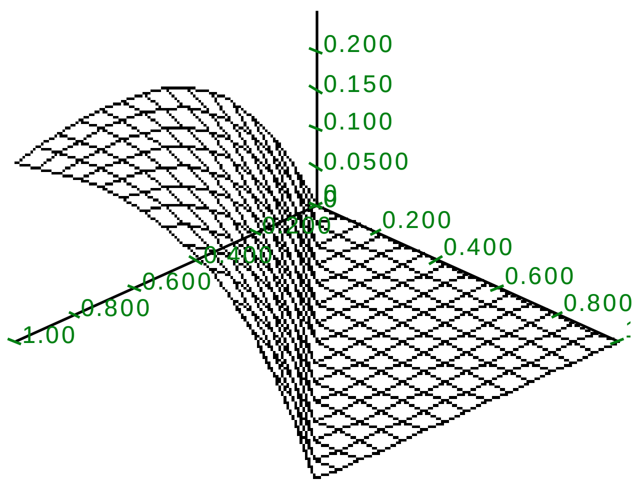
But this is in N. Thus, A is a compact, self adjoint operator on the Hilbert space N. From the above, it has an eigenvalue μ with μ < ||A||. We now make M_2 and continue the process. The previous Theorem 32 establishes that all eigenvalues of A will be found this away.

Remark The above Theorem 31 requires that A be compact and self-adjoint. One cannot guarantee non-trivial eigenvalues and eigenfunctions if A is not self-adjoint. An example follows:

$$\text{Let } K(x,y) = \begin{cases} 0 & \text{if } 0 \le x \le y \le 1 \\ e^{y-x} - e^{2(y-x)} & \text{if } 0 \le y \le x \le 1 \end{cases}$$

and H(x,y) = K(y,x).

Here is the graph of K:



Graph of K as given above.

Define $A(f)(x) = \int_0^1 K(x,y) f(y)\, dy$ **and** $B(g)(x) = \int_0^1 H(x,y) g(y)\, dy$. **Each of A and B is a bounded, compact linear function from $L^2[0,1]$ to $L^2[0,1]$. Also A* = B. Moreover, these are equivalent:**

(a) g = A(f), and

(b) $g'' + 3g' + 2g = f$, $g(0) = g'(0) = 0$.

Also, these are equivalent:

(c) g = B(f), and

(d) $g'' - 3g + 2g = f$, $g(1) = g'(1) = 0$.

To see that A has no non-trivial eigenvalues and eigenfunctions, note that $A(f)(x) = \int_0^x K(x,y)\, f(y)\, dy$, so that A is a Volterra Integral Operator -- the integral is from 0 to x, not 0 to 1. If there were a number $\lambda \neq 0$ and a function f such that

$$\lambda f(x) = \int_0^x K(x,y)\, f(y)\, dy$$

then $|f(x)| \leq \frac{1}{|\lambda|^n} B^n \frac{x^n}{n!} M$, for each integer n, where $|K(x,y)| \leq B$ and $|f(x)| \leq M$. Hence $||f|| = 0$.

Remark Here is a pair of self adjoint linear operators that are not compact: $A(x) = x$ for all x in any infinite dimensional Hilbert space, and $B(f)(t) = t\, f(t)$ in $L^2([0,1])$. A way to see the latter one is not compact is to note that it has no eigenvalues.

Assignment

(17.1) Go back and see where the proof of Theorem 31 uses $A = A^*$.

(17.2) Suppose that E has a countable maximal orthonormal sequence. If P is a continuous projection then these are equivalent:

(a) P is a compact operator, and
(b) The range of P is finite dimensional.

(17.3) Definition: Suppose that A is a bounded, self-adjoint operator on E and that $\{\varphi_p\}_{p=1}^{\infty}$ is a maximal orthonormal family. Then $\text{trace}A = \sum \langle A\varphi_n, \varphi_n \rangle$.

This sum may be "∞". If λ_n is a sequence of real numbers, E is $L^2[0,1]$ and

$$K(s,t) = \sum \lambda_n\, \varphi_n(s)\, \varphi_n(t) \quad \text{and} \quad A(f)(x) = \int_0^1 K(x,y)\, f(y)\, dy$$

then $\text{trace}A = \int_0^1 K(s,s)\, ds$.

(17.4) Show that if $K(s,t) = 4\cos(s-t)$ then the operator given by

$$A(f)(x) = \int_{-\pi}^{\pi} K(x,s)\, f(s)\, ds$$

is a self-adjoint operator on $L^2[-\pi,\pi]$ with finite dimensional range. Find projections and numbers such that the operator is given by

$$\lambda_1 P_1 + \lambda_2 P_2 + \quad + \lambda_n P_n.$$

(17.5) Let $\{\varphi_p\}_{p=1}^{\infty}$ be a maximal orthonormal family and A be defined by Ax $= \sum_{p=1}^{\infty} \frac{1}{p} \langle x, \varphi_p \rangle \varphi_{p+1}$ - so that A is a "weighted shift". Show that

(a) $\|A\|^2 \le \sum_{p=1}^{\infty} \frac{1}{p^2}$.

(b) $A = \lim_n \sum_{p=1}^{n} \frac{1}{p} \langle \ , \varphi_p \rangle \varphi_{p+1}$ in BLT.

(c) A is compact.

(d) A has no nonzero eigenvalues.

(e) Give a formula for A*.

MAPLE remarks

Here is the syntax for drawing the graph of a kernel. Seeing the shape of the graph gives a geometric understanding to the algebraic notion of symmetry: K(x,y) = K(y,x). The following graph kernel is not symmetric.

```
> K:=proc(x,y) if x <= y then 0
        else exp(y-x) - exp(2*(y-x))
        fi end;
> plot3d(K, 0..1, 0..1,axes=NORMAL);
```

Sometimes a finite dimensional analogue of an infinite dimensional result will show that the ideas are quite easy indeed. Consider this example in connection with Assignment 17.5.

```
> with(linalg):
> A:=array([[0,0,0,0],[1,0,0,0],[0,1/2,0,0],[0,0,1/3,0]]);
> evalf(norm(A,1));evalf(norm(A,2));evalf(norm(A,infinity));
> eigenvals(A);
> transpose(A);
```

Section 18: Normal Operators and The More General Paradigm

Self adjoint operators play the role in BLT that real numbers do in $\mathbb{C}$ -- not only in the sense that A = A*, but also, if T is in BLT, then there are self-adjoint operators A and B such that T = A + iB. In fact A = $\frac{T+T^*}{2}$ and B = $\frac{1}{2i}$ (T-T*).

We continue our investigation into the representation of operators as $\sum \lambda_p P_p$. For reasons explained previously, we ask that the sequence of projections should form a resolution of the identity.

Definition T is *normal* provided that TT* = T*T.

Theorem 34 Suppose that T is in BLT and each of A and B is self adjoint with T = A + iB. These are equivalent:
(a) T is normal, and
(b) AB = BA.

Proof: To prove (b), simply compute AB and BA. To prove (a), compute TT* and T*T.

Theorem 35 With the supposition of Theorem 34, suppose also that T is normal. These are equivalent:
(a) T is compact,
(b) Each of A and B is compact, and
(c) T* is compact.

Proof
(a) (b) $|Tx|^2 = |(A+iB)x|^2$

$$= |Ax|^2 + i < Bx, Ax> - i <Ax, Bx > + |Bx|^2$$

$$\geq \begin{cases} |Ax|^2 \\ |Bx|^2 \end{cases}$$

(b) (a) $|Tx_n - Tx_m|^2 = |Ax_n - Ax_m|^2 + |Bx_n - Bx_m|^2$.

Theorem 36. Suppose that T is a compact, normal operator. Then T has an eigenvalue λ with max(||A||, ||B||) ≤ |λ|.

Proof: We know there is an eigenvalue λ for A with |λ| = ||A||. Consider {x: Ax = λx} = N(λI-A). This is a Hilbert space. To see that B maps N into N, let n N. Then ABn = BAn = Bλn = λBn. Also, B is compact and

selfadjoint on N. There is μ such that $\mu y = By$. Let $\sigma = \lambda + i\mu$ and $v = y$. Then $Tv = (A+iB)v = Av + iBv = \lambda v + i\mu v = \sigma v$. Also, $|\sigma| \geq \max(|\lambda|, |\mu|)$.

Theorem 37 **If T is bounded and normal, then**
(a) $|Tx| = |T^*x|$,
(b) If $Tx = \lambda x$ then $T^*x = \lambda^* x$, and
(c) If $\lambda \neq \mu$, $Tx = \lambda x$, and $Ty = \mu y$, then $\langle x, y\rangle = 0$.

Remark **Actually, statement (a) is equivalent to the statement that T is normal.**

Theorem 38 **If T is a compact normal operator on E then there is a family $\{\varphi_p\}$ of orthonormal vectors which is maximal in E and a sequence $\{\lambda_p\}_{p=1}^{\infty}$ of complex numbers such that, if x is in E, then**

$$Tx = \sum_{p=1}^{\infty} \lambda_p \langle x, \varphi_p\rangle \varphi_p.$$

Moreover, if 0 is not an eigenvalue of T, then the eigenvectors form a complete orthonormal system.

The proof of this theorem is just like that of Theorem 36. The proof that the orthonormal family is maximal, or complete, can be argued as follows: suppose that v is a nonzero vector with $\langle v, \varphi_p\rangle = 0$ for all p. Then v must be in the nullspace of A so that $Av = 0v$. This contradicts the supposition that 0 is not an eigenvalue of A.

Assignment

(18.1). Suppose that $\{\lambda_p\}_{p=1}^{\infty}$ is a bounded sequence of complex numbers and $\{P_p\}_{p=1}^{\infty}$ is a sequence of orthogonal projections. Show that $\sum \lambda_p P_p$ is normal.

(18.2). Show that $T(x) = \sum \langle x, \varphi_p\rangle \varphi_{p+1}$ is not normal.

(18.3). Show that $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ is a projection that is *not* normal.

(18.4). Give an example of a matrix that is normal but not self adjoint.

MAPLE Remark **The challenge is to make up a normal operator that is not self-adjoint.**

```
> with(linalg):
> T:=array([[0,0,0,0],[1,0,0,0],[0,1/2,0,0],[0,0,1/3,0]]);
> A:= evalm(A+transpose(A))/2;
> B:= evalm(T-transpose(T))/(2*I);
```

```
> evalm(A &* B);evalm(B &* A);
```

Since A and B do not commute, T is not normal.

On the other hand here is a matrix that is normal, but not self adjoint:

$$T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

```
> T:=array([[1,-1],[1,1]]);
> A:=evalm((T+transpose(T))/2);
> B:=evalm((T-transpose(T))/(2*I));
```

Note that A and B are self-adjoint. This is clear for A. To see that it is true for B, compute:

```
> Bstar:=transpose(-B);
> evalm(A &* B); evalm(B &* A);
> evalm(A + I*B);
```

Is it clear from this that T is normal?