Section 21: The Fredholm Alternative Theorems

The Fredholm Alternative theorems concern the equation

$$(1-A)u = f.$$

These ideas come up repeatedly in differential equations and in integral equations. The Alternative Theorems state necessary and sufficient conditions for the equation (1-A)u = f to have a solution u for some previously specified f. There are two alternatives: either the equation has exactly one solution for all f or the equation has many solutions for some f's and none for others. Those for which there are solutions are characterized.

Lemma 43. Suppose that A is compact. There are orthonormal sequences $\{\Phi_p\}$ and $\{\Theta_p\}$ such that $[1-A] = \sum_{p=1}^{\infty} (1-\lambda_p)< x, \varphi_p> \Theta_p$.

<u>Suggestion of Proof:</u> $- A^*- A + A^*A$ is compact and self adjoint. Also,

$$[1-A]^*[1-A] = 1 - A^*- A + A^*A.$$

Let $\quad - A^*- A + A^*A = \Sigma_p - \mu_i < \ ,\varphi_i>\varphi_i.$

Also, $\quad I = \Sigma_p< \ ,\varphi_i>\varphi_i$

so that $\quad [1-A]^*[1-A] = \Sigma_p (1-\mu_i) < \ ,\varphi_i>\varphi_i.$

Let $\quad \lambda_i = 1-\sqrt{1-\mu_i}$ **and** $\theta_i = [1-A](\varphi_i)/(1-\lambda_i).$

Note that $\quad < \theta_i, \theta_j > = 0.$

Then $\quad x = \Sigma_p< x, \varphi_i >\varphi_i$ **and** $\quad [1-A](x) = \sum_{p=1}^{\infty} (1-\lambda_p)< x, \varphi_p> \Theta_p.$

Theorem 44 Suppose that A is compact. Exactly one of the following alternatives holds:
(a) if f is in E then the equation (1-A)u = f has only one solution,
(b) the equation (1-A)u = 0 has more than one solution.

Theorem 45 If A is compact and the first alternative holds for the equation (1-A)u = f then it also holds for the equation (1-A*)u = f.

Theorem 46 Suppose A is compact and the second alternative holds for the equation. Then these are equivalent:
(1) the equation (1-A)u = f has a solution, and
(2) < f , z > = 0 for all solutions z for the equation (1-A*)z = 0.

(Hint: to see that 2 implies 1, write out what is the null space for (1-A*).)

Remark For general linear operators, the question might be, given f, does the equation Bu = f have a solution. For square matrices, we can completely answer the question:

Exactly one of the following holds:
(a) $\det(B) \neq 0$ and if f is in E then Bu = f has exactly one solution.
(b) $\det(B) = 0$ and Bu = 0 has more than one solution.

For the general compact operator, the alternatives are not definitive: Let

$$B = \sum_{p=1}^{\infty} \frac{1}{p} \langle \ , \varphi_p \rangle \varphi_p.$$

Then B(u) = 0 has only one solution, but there are f's for which B(u) = f has no solution.

Assignment
(21.1) In R^2, let A(x,y) = {x+y, 2y}. Show that (1-A)u = 0 and (1-A*)u = 0 have non-trivial solutions. What conditions must f satisfy in order that (1-A)u = f should have a solution?

(21.2) In $L^2[0,1]$, let $A(f)(x) = \int_0^1 [1+\cos(\pi x)\sin(\pi t)]\, f(t)\, dt$. Show that (1-A)u = 0 and (1-A*)u = 0 have non-trivial solutions. If (1-A)u = f has a solution, what conditions must f satisfy?

(21.3) Show that if A is compact and normal then the representation can be

$$[1-A] = \sum_{p=1}^{\infty} (1-\lambda_p)\langle x, \varphi_p \rangle \varphi_p.$$

MAPLE Remark:
In the matrix situation, the Fredholm Alternative Theorems are no more interesting than evaluation of the determinant. If the determinant is not zero, the matrix is in the first alternative. It the determinant is zero, it is in the second alternative. Since

$$\det(1-A) = \det(1-A^*),$$

it is no surprise that the first alternative holds for 1-A if and only if it holds for 1-A*. The last of the Fredholm Alternative theorems is a little more interesting. We examine it with a matrix for which det(1-A) = 0, so that the second alternative holds.

```
> with(linalg):
> A:=array([[0,-2,1],[-2,-5,-2],[-3,-4,8]]);
> B:=evalm(diag(1,1,1)-A);
> det(B);
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Because det(1-A) = 0, this A is in the second alternative. To determine for which f's the equation

(1-A)u = f

has a solution u, we find the null space of 1-A*.

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> nullspace(evalm(diag(1,1,1)-transpose(A)));
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Ask what vectors are orthogonal to this one. There is a plane of them:

a [1,5,0] + b [1,0,5].

We solve (1-A)u = [1,5,0] and (1-A) = [1,0,5].

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> solve({x+2*y-z=1,2*x+6*y+2*z=5,3*x+4*y-7*z=0},{x,y,z});
> solve({x+2*y-z=1,2*x+6*y+2*z=0,3*x+4*y-7*z=5},{x,y,z});
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Finally, we predict the equation (1-A)u = [1,5,1] has no solution.

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> solve({x+2*y-z=1,2*x+6*y+2*z=5,3*x+4*y-7*z=1},{x,y,z});
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Section 22: Closed Operators

We change subjects now. The subject is continuity. The first notion of continuity encountered in this course was approached from the perspective of sequences and has been restricted to linear functions. Bounded linear functions and unbounded linear functions are words which were used to characterize continuous and non-continuous linear functions. Later, a notion was investigated that was more restrictive than contiuunous. It is that of a compact operator. Thus, we had these three: compact operator, bounded operator, and unbounded operator. Now we introduce another that lies among these. It is that of a closed operator.

Definition A function is *closed* provided its graph is closed -- in the sense that if $\{x_p\}$ has limit y and $\{L(x_p)\}$ has limit z then y is in the domain of L and L(y) = z.

Examples
(1) Let f(x) be the function from R to R defined by f(x) = 1/x if x is not zero and f(0) = 0. This function is closed.
(2) Let f(x) be the function from R to R defined by f(x) = x/|x| if x is not zero and f(0) = 0. This function is not closed.

We have characterized maps which satisfy the paradigm and which are compact, bounded, and unbounded depending on the character of their eigenvalues. Here is the situation for closed operators:

Theorem 47. Suppose that $A = \Sigma_p \lambda_p < \ , \varphi_p> \varphi_p$ and that the domain is as large as possible in the sense that

$$D(A) = \{ x: \Sigma_p |\lambda_p|^2 |< x, \varphi_p>|^2 < \infty\}$$

It follows that the operator A is closed.

<u>Suggestion of proof.</u> Suppose that $\lim_n x_n = y$ and $\lim_n A(x_n) = z$. We want to show that $\Sigma_p |\lambda_p|^2 |< y, \varphi_p>|^2 < \infty$ and that $\Sigma_p \lambda_p <y, \varphi_p> \varphi_p = z$. For each N,

$$0 = < \lim_n \Sigma_p \lambda_p <x_n, \varphi_p> \varphi_p,\varphi_N> - < z, \varphi_N>$$
$$= \lim_n \lambda_N < x_n, \varphi_N> - < z, \varphi_N>$$
$$= \lambda_N < y, \varphi_N> - < z, \varphi_N>.$$

Hence, $\Sigma_p |\lambda_p|^2 |< y, \varphi_p>|^2 = \Sigma_p|< z, \varphi_p>|^2 < \infty$, from which we conclude that y is in the domain of A and z = A(y).

Remark
(1) To see the generality of a closed operator, note that if A is a symmetric, densely defined operator in a Hilbert space, then there is a closed, symmetric operator B such that A is contained in B.
(2) Because of the previous remark, we might as well always take a symmetric, densely defined operator to be closed. While this will not imply

that the operator is continuous, there is this idea: It is always possible to redefine the inner-product on D(A) such that the domain of A becomes a Hilbert space and A becomes a bounded operator on the domain of A, In fact, for x and y in the domain of A, define the new inner-product by

$$< x, y>_n = < x, y> + < Ax, Ay>.$$