Section 23: The Deficiency of A

We have examined the paradigm whereby A can be written as

(23.1) $$A(x) = \sum_{p=1}^{\infty} \lambda_p < x , \phi_p > \theta_p$$

in case $\{\lambda_p\} \to 0$ or in case $\{\lambda_p\} \to \infty$. We now examine the special case that $\{\frac{1}{\lambda_p}\}$ is bounded.

Remark: It goes without saying that we are supposing that no λ_p is zero. It may be that $\{ \phi_p\}$ is not a maximal orthonormal family. Then A could be extended to all of E by making the value zero at the extension of the ϕ_p's to a maximal orthonormal family. In that case, we can characterize the null space of this extension of A:

$$N(A) = \{ x: \ < x, \phi_p > = 0 \text{ if } \lambda_p \neq 0 \}.$$

Theorem. Suppose that A can be written in the paradigm of (23.1) and that R(A) represents the range of A. Then, these are equivalent:

(1) $\{\frac{1}{\lambda_p}\}$ is a bounded sequence,

(2) $R(A) = \{y: y = \Sigma_{\lambda_p \neq 0} < y , \theta_p > \theta_p$.

(3) R(A) = *perp*(N(A*)), where *perp*(N(A*)) is the collection of vectors perpendicular to the nullspace of A*.

(4) R(A) is closed.

<u>Suggestions for a proof.</u>

1 => 2. Suppose (1). Let

$$S = \{ y: y = \ \Sigma_{\lambda_p \neq 0} < y , \theta_p > \theta_p\}.$$

It is clear that the range of A is contained in S. To see that the range of A is S, suppose y is in S. Let

$$x = \ \Sigma_{\lambda_p \neq 0} \ \frac{1}{\lambda_p} < y, \theta_p > \phi_p.$$

Because of (1), x is in E. Also, Ax = y.

2 => 3. Recall that

$$A^*(z) = \sum_{p=1}^{\infty} \lambda_p^* < z, \theta_p > \phi_p .$$

Also, as in the Remark above

(23.2) $$N(A^*) = \{z: < z, \theta_p > = 0 \text{ if } \lambda_p \neq 0 \}.$$

And,

$$perp(N(A^*)) = \{w: < w, \theta_p > = 0 \text{ if } \lambda_p = 0\}.$$

First we show that *perp*(N(A*)) is contained in R(A). Suppose that w is in *perp*(N(A*)). Then

$$w = \Sigma_p < w , \theta_p > \theta_p = \Sigma_{\lambda_p \neq 0} < w, \theta_p > \theta_p.$$

To see that this w is in R(A), we need to produce x so that A(x) = w. As above, choose x =

$$x = \Sigma_{\lambda_p \neq 0} \frac{1}{\lambda_p} < y, \theta_p > \phi_p.$$

As before Ax = w. Thus, *perp*(N(A*)) is contained in R(A).

Finally, we show that R(A) is contained in *perp*(N(A*)). Suppose that y is in R(A) and z is in N(A*). Then y is a combination of the θ_p's

$$< y, z > = < \Sigma_{\lambda_p \neq 0} < y, \theta_p > \theta_p , z > = \Sigma_{\lambda_p \neq 0} < y, \theta_p > < \theta_p , z > = 0$$ by equation (23.2).

Thus, y is in *perp*(N(A*)), completing the outline for 2 => 3.

(3) = (4) This follows because *perp*(N(A*)) is closed.

(4) => (1) Suppose that $\{1/\lambda_p\}$ is unbounded. Then

$$A^{(-1)} = \sum_p \frac{1}{\lambda_p} < z , \theta_p > \phi_p$$

is an unbounded operator. Consequently, $A^{(-1)}$ cannot have domain all of E. Thus, there is z in E so that z is not in the domain of $A^{(-1)}$ or, what is the same, z is not in the R(A). Let y_n be defined as follows:

$$y_n = \sum_{p=1}^{n} \frac{1}{\lambda_p} < z, \theta_p > \phi_p.$$

Then, $A(y_n)$ is in the range of A and is seen to be

$$A(y_n) = \sum_{p=1}^{n} 1 < z, \theta_p > \theta_p.$$

Consequently, $A(y_n)$ is a sequence in R(A) that has limit z which is not in R(A). Thus, R(A) is not closed. This is a contradiction.

This finishes an outline for a proof of the Theorem.

Definition. There are two numbers associated with A:

$\eta(A)$ = the dimension of the kernel of A.

and

$\delta(A)$ = the dimension of the kernel of A*.

We call the number $\delta(A)$ the *deficiency of A*. In a sense, this number measures how much the range of A is deficient in filling E.

Examples:

1. If A is given by the simple paradigm,

$$A(x) = \sum_{p=1}^{\infty} \lambda_p < x , \phi_p > \phi_p ,$$

then $\eta(A)$ is the number of λ_p 's such that $\lambda_p = 0$, and $\delta(A) = \eta(A)$.

2. If A is the right shift operator

$$A(x) = \sum_{p=1}^{\infty} \lambda_p < x , \phi_p > \phi_{p+1} ,$$

so that

$$A^*(x) = \sum_{p=1}^{\infty} \lambda_p^* < x , \phi_{p+1} > \phi_p ,$$

then $\eta(A) = 0$ and $\delta(A) = 1$.

3. We compute $\eta(A)$ and $\delta(A)$ for

$$\mathbf{K}[f](x) = \int_0^1 \cos(\pi\,(x-y))\, f(y)\, dy .$$

First, imagine **K** in the simple paradigm noting that the range of **K** is two dimensional and

$$\mathbf{K}[\cos(\pi\, y)](x) = \frac{\cos(\pi\, x)}{2} \quad \text{and} \quad \mathbf{K}[\sin(\pi\, y)](x) = \frac{\sin(\pi\, x)}{2} .$$

Hence, $\eta(A)$ in $L^2[0, 1]$ is ∞ . Also, $\delta(A) = \infty$, since **K** = **K***.

4. We compute $\eta(\mathbf{K})$ for

$$\mathbf{K}[f](x) = x\int_0^x f(y)\,dy - \int_0^x y\,f(y)\,dy = \int_0^x (x-y)\,f(y)\,dy$$

Suppose f is in the kernel of **K**. Then **K**[f](x) = 0. Taking the derivative with respect to x, we see that

$$0 = \frac{\partial}{\partial x}\,\mathbf{K}[f](x) = \int_0^x f(y)\,dy,$$

and

$$0 = \frac{\partial^2}{\partial x^2}\,\mathbf{K}[f](x) = f(x).$$

Thus, f is zero and the null space of A must be zero. Hence $\eta(A) = 0$.