Section 23: The Deficiency of A

We have examined the paradigm whereby A can be written as

\[ A(x) = \sum_{p=1}^{\infty} \lambda_p < x, \phi_p > \theta_p \]

in case \( \{ \lambda_p \} \to 0 \) or in case \( \{ \lambda_p \} \to \infty \). We now examine the special case that \( \{ \frac{1}{\lambda_p} \} \) is bounded.

Remark: It goes without saying that we are supposing that no \( \lambda_p \) is zero. It may be that \( \{ \phi_p \} \) is not a maximal orthonormal family. Then A could be extended to all of E by making the value zero at the extension of the \( \phi_p \)'s to a maximal orthonormal family. In that case, we can characterize the null space of this extension of A:

\[ N(A) = \{ x: < x, \phi_p > = 0 \text{ if } \lambda_p \neq 0 \} \]

Theorem. Suppose that A can be written in the paradigm of (23.1) and that R(A) represents the range of A. Then, these are equivalent:

1. \( \{ \frac{1}{\lambda_p} \} \) is a bounded sequence,
2. \( R(A) = \{ y: y = \sum_{\lambda_p \neq 0} < y, \theta_p > \theta_p \} \).
3. \( R(A) = \text{perp}(N(A^*)) \), where \( \text{perp}(N(A^*)) \) is the collection of vectors perpendicular to the nullspace of A*.
4. \( R(A) \) is closed.

Suggestions for a proof:

1 => 2. Suppose (1). Let

\[ S = \{ y: y = \sum_{\lambda_p \neq 0} < y, \theta_p > \theta_p \} \].

It is clear that the range of A is contained in S. To see that the range of A is S, suppose y is in S. Let

\[ x = \sum_{\lambda_p \neq 0} \frac{1}{\lambda_p} < y, \theta_p > \phi_p \].

Because of (1), x is in E. Also, Ax = y.

2 => 3. Recall that
\[ A^*(z) = \sum_{p=1}^{\infty} \lambda_p \cdot \langle z, \theta_p \rangle \cdot \phi_p. \]

Also, as in the Remark above

(23.2) \[ N(A^*) = \{ z: \langle z, \theta_p \rangle = 0 \text{ if } \lambda_p \neq 0 \}. \]

And,

\[ \text{perp}(N(A^*)) = \{ w: \langle w, \theta_p \rangle = 0 \text{ if } \lambda_p = 0 \}. \]

First we show that \( \text{perp}(N(A^*)) \) is contained in \( R(A) \). Suppose that \( w \) is in \( \text{perp}(N(A^*)) \). Then

\[ w = \sum_p < w, \theta_p > \cdot \lambda_p = \sum_{\lambda_p \neq 0} < w, \theta_p >. \]

To see that this \( w \) is in \( R(A) \), we need to produce \( x \) so that \( A(x) = w \). As above, choose \( x = \sum_{\lambda_p \neq 0} \frac{1}{\lambda_p} < y, \theta_p > \cdot \theta_p. \)

As before \( Ax = w \). Thus, \( \text{perp}(N(A^*)) \) is contained in \( R(A) \).

Finally, we show that \( R(A) \) is contained in \( \text{perp}(N(A^*)) \). Suppose that \( y \) is in \( R(A) \) and \( z \) is in \( N(A^*) \). Then \( y \) is a combination of the \( \theta_p \)'s

\[ < y, z > = \sum_{\lambda_p \neq 0} < y, \theta_p > \cdot \lambda_p = \sum_{\lambda_p \neq 0} < y, \theta_p > < \theta_p, z > = 0 \] by equation (23.2).

Thus, \( y \) is in \( \text{perp}(N(A^*)) \), completing the outline for \( 2 \Rightarrow 3 \).

(3) = (4) This follows because \( \text{perp}(N(A^*)) \) is closed.

(4) \( \Rightarrow \) (1) Suppose that \( \{ 1/\lambda_p \} \) is unbounded. Then

\[ A^{(-1)} = \sum_p \frac{1}{\lambda_p} < z, \theta_p > \cdot \phi_p \]

is an unbounded operator. Consequently, \( A^{(-1)} \) cannot have domain all of \( E \). Thus, there is \( z \) in \( E \) so that \( z \) is not in the domain of \( A^{(-1)} \) or, what is the same, \( z \) is not in the \( R(A) \). Let \( y_n \) be defined as follows:

\[ y_n = \sum_{p=1}^{n} \frac{1}{\lambda_p} < z, \theta_p > \cdot \phi_p. \]

Then, \( A(y_n) \) is in the range of \( A \) and is seen to be

\[ A(y_n) = \sum_{p=1}^{n} 1 < z, \theta_p > \cdot \theta_p. \]
Consequently, \( A(y_n) \) is a sequence in \( \text{R}(A) \) that has limit \( z \) which is not in \( \text{R}(A) \). Thus, \( \text{R}(A) \) is not closed. This is a contradiction.

This finishes an outline for a proof of the Theorem.

**Definition.** There are two numbers associated with \( A \):

\[
\eta(A) = \text{the dimension of the kernel of } A.
\]

and

\[
\delta(A) = \text{the dimension of the kernel of } A^*.
\]

We call the number \( \delta(A) \) the *deficiency of \( A \).* In a sense, this number measures how much the range of \( A \) is deficient in filling \( E \).

**Examples:**
1. If \( A \) is given by the simple paradigm,
   \[
   A(x) = \sum_{p=1}^{\infty} \lambda_p < x, \phi_p > \phi_p ,
   \]
then \( \eta(A) \) is the number of \( \lambda_p \)'s such that \( \lambda_p = 0 \), and \( \delta(A) = \eta(A) \).

2. If \( A \) is the right shift operator
   \[
   A(x) = \sum_{p=1}^{\infty} \lambda_p < x, \phi_p > \phi_{p+1} ,
   \]
so that
   \[
   A^*(x) = \sum_{p=1}^{\infty} \lambda_p^* < x, \phi_{p+1} > \phi_p ,
   \]
then \( \eta(A) = 0 \) and \( \delta(A) = 1 \).

3. We compute \( \eta(A) \) and \( \delta(A) \) for
   \[
   K[f](x) = \int_{0}^{1} \cos(\pi (x-y)) f(y) \, dy.
   \]
First, imagine \( K \) in the simple paradigm noting that the range of \( K \) is two dimensional and
   \[
   K[\cos(\pi y)](x) = \frac{\cos(\pi x)}{2} \text{ and } K[\sin(\pi y)](x) = \frac{\sin(\pi x)}{2}.
   \]
Hence, \( \eta(A) \) in \( L^2[0, 1] \) is \( \infty \). Also, \( \delta(A) = \infty \), since \( K = K^* \).
4. We compute $\eta(K)$ for

$$K[f](x) = x \int_0^x f(y) \, dy - \int_0^x y f(y) \, dy = \int_0^x (x - y) f(y) \, dy$$

Suppose $f$ is in the kernel of $K$. Then $K[f](x) = 0$. Taking the derivative with respect to $x$, we see that

$$0 = \frac{\partial}{\partial x} K[f](x) = \int_0^x f(y) \, dy,$$

and

$$0 = \frac{\partial^2}{\partial x^2} K[f](x) = f(x).$$

Thus, $f$ is zero and the null space of $A$ must be zero. Hence $\eta(A) = 0$. 

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