Section 3: Self Adjoint Transformations in Inner-Product Spaces

In these notes, we work in an *inner-product space*. Often, we consider the space E as a collection of vectors over the complex (or real) numbers. We take an *inner product*, or *dot-product*, to be defined on ExE and assume the inner product to have these properties:

$$< x, y > = < y, x>^*$$

$$< x+\alpha y, z > = < x, z > + \alpha < y, z>,$$

and $$< x, x> \; > \; 0 \text{ if } x \neq 0.$$

Examples: Some innerproduct spaces every young analysists should know are denoted R^n, $R^\infty = {}_n R^n$, L^2, and $L^2[0,1]$. We take R^{n+1} R^n in a cannonical way. Every innerproduct space is a normed space for take the norm to be defined by

$$|x| = \sqrt{<x,x>}\,.$$

One should note that not every normed space arises from an innerproduct space. In fact, a characterization of those norms which arise from an inner product is that they satisfy the parallelogram identity.

The difference in a normed space and an innerproduct space is an important notion. It would be well to have some examples of innerproduct spaces and normed spaces to build intution. We present examples in class.

The best known inequality in an inner product space is the Cauchy Schwartz inequality.

Theorem 6: If x and y are in E then $|< x, y>| \leq |x| \; |y|$.
<u>Hint for proof:</u> Consider $h(t) = x + <x,y> y\, t$ and $|h(t)|^2$.

Linear transformations that are self adjoint have a simple structure. We begin with these.

Definition In an inner product space {E,< , >}, the linear transformation A on E is *self adjoint* if <Ax,y> = <x,Ay> for all x and y in the domain of A.

Examples: Here are three examples to remind you that the notion of "self-adjoint" is pervasive.

(a) If A is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then A is self-adjoint in R^2 with the usual dot product.

(b) If K(f)(x) is $\int_0^1 \cos(x-y)\, f(y)\, dy$ then K is self-adjoint on $L^2[0,1]$ with the usual dot product.

(c) If $A(f) = (pf\,')' + qf$, with p and q smooth, then A is self-adjoint on the collection of functions in $L^2[0,1]$ having two derivatives and that are zero at 0 and at 1. A useful identity to understand this example is

$$(pf\,')'g - f(pg')' = [\,p(f\,'g - fg')]'.$$

Definition: Numbers λ and vectors v are *eigenvalues* and *eigenvectors,* respectively, for the linear transformation A if $Av = \lambda v$.

Theorem 7: If A is self-adjoint then all eigenvalues are real and eigenvectors corresponding to different eigenvalues are orthogonal.

<u>Suggestion of Proof.</u> Suppose that $x \neq 0$ and that $Ax = \lambda x$. We hope to show that $\lambda = \lambda^*$:

$$\lambda\langle x,x\rangle = \langle Ax,x\rangle = \langle x,Ax\rangle = \lambda^*\langle x,x\rangle.$$

Finally $(\lambda-\mu)\langle x,y\rangle = \langle Ax,y\rangle - \langle x,Ay\rangle = 0$, so that $\langle x,y\rangle = 0$.

Corollary 8: In the decomposition of Theorem 1, if A is self adjoint then each $N_i = 0$.

<u>Suggestion of a proof.</u> Let m be the smallest integer such that $N^m = 0$ and suppose that $m \neq 1$. Then, for each x in E and use the fact that each N is a polynomial in A to see that N is self adjoint if A is. Consequently,

$$\langle N^{m-1}x,\ N^{m-1}x\rangle = \langle N^{m-2}x,\ N^{m}x\rangle = 0.$$

Assignment

(3.1) Give a point in L^2 that is not in R^∞.

(3.2) Construct a proof of the Cauchy Schwartz inequality in a general inner product space.

(3.3) Show that

$$\begin{pmatrix} -.5 & .5 \\ .5 & -.5 \end{pmatrix}$$

is self adjoint with the usual dot product. Find its Jordan Form. Repeat this for

$$\begin{pmatrix} -.5 & .5 & 0 \\ .5 & -.5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(3.4) Suppose that A is self-adjoint and has spectral resolution given by

$$A = \sum_{i=1}^{k} \lambda_i P_i.$$

Suppose also that $\lambda_1 = 0$ and all other λ's are not zero. Recalling that if e is in E then

$$e = \sum_{i=1}^{k} P_i(e),$$

show that $P_1(e)$ is in the nullspace of A and that

$$\sum_{i=2}^{k} P_i(e)$$

is in the range of A. In fact, find u such that

$$A(u) = \sum_{i=2}^{k} P_i(e).$$

(3.5) Find the eigenvalues and eigenvectors for the matrices in the Assignment of Section 1. Relate this to the Jordan Form for self adjoint matrices.

(3.6) Verify that

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are orthogonal. Create A such that these three vectors are eigenvectors and 0,-1, and 2 are eigenvalues for A. (Hint:

$$Ax = 0 \langle x,v_1\rangle v_1/|v_1| - 1 \langle x, v_2\rangle v_2 /|v_2| + 2 \langle x, v_3\rangle v_3 /|v_3| .)$$

(3.7) Let

$$K[f](x) = \int_0^1 \cos(\pi(x-y))\, f(y)\, dy$$

on $L^2[0,1]x[0,1]$. Find the eigenvalues and eigenfunctions for K. Show that the eigenfunctions are orthogonal.

Section 4: The Gerschgorin Circle Theorem

If all the eigenvalues of A are in the left-half plane -- have negative real part -- then solutions for

$$Z' = AZ$$

collapse to zero. If at least one eigenvalue is in the right-half plane, then some solution grows without bound. Consequently, it is very important to know where the eigenvalues of A are, even if the matrix is such a mess that the eigenvalues are hard to fine. You can guess that people have turned their attention to locating eigenvalues. The next theorem is a classical result. More recent results are included in this section as a cited reference.

Theorem 9 (Gerschgorin's Circle Theorem) If A is a matrix and

$$S = \ {}_m\{z: |A_{mm}-z| \le \sum_{j\neq m} |A_{jm}|\},$$

then every eigenvalue of A lies in S.

<u>Suggestion for Proof.</u>
Lemma: If A is a matrix and

$$|A_{mm}| > \sum_{i\neq m} |A_{im}|$$

for all m, then A is invertible.
Here's how to see this: Suppose A is not invertible. Thus 0 = det(A) = det(A^T). Let x be such that $A^T x = 0$ and $x \neq 0$. Let m be such that $|x_m| = \max_i |x_i|$. Then

$$0 = (A^T x)_m = \sum_i A^T{}_{mi}\, x_i.$$

Or, $$A_{mm}x_m = -\sum_{i\neq m} A^T{}_{mi}x_i = -\sum_{i\neq m} A_{im}x_i.$$

Hence, $$|A_{mm}|\,|x_m| = |\sum_{i\neq m} A_{im}x_i| \le |x_m| \sum_{i\neq m} |A_{im}|.$$

<u>Proof of Theorem 9</u>: Suppose that λ is an eigenvalue of A not in S. Then

$$|\lambda - A_{mm}| > \sum_{i\neq m} |A_{im}|.$$

Thus, det(λI-A) ≠ 0 and λ can not be an eigenvalue. This is a contradiction.

Remark: It is easy to see that the Gershgorin Circle Theorem might be important for determining the behavior of systems of differential equations. For example, as noted in the beginning of this section, if all eigenvalues of the matrix A are in the left half of the complex plane, then solutions of the differential equation

$$Z' = AZ$$

have asymptotic limit zero.

As a result of the importance of these ideas, considerable work has been done to improve the bounds. One improvement can be found in the following:

David G. Feingold and Richard S. Varga, *Block Diagonally Dominant Matrices and Generalizations of the Gerschgorin Circle Theorem*, Pacific J. Math., 12 (1962) 1241-1249. (QA1.P3.)

In that paper, the authors consider block matrices down the diagonal and state the inequalities in terms of a *matrix norm*.

Definition: If A is a matrix on R^n with norm $|x| = \sqrt{\Sigma_i x_i^2}$, then the *2-norm of A* is defined by

$$||A||_2 = \sup\left(\frac{|Ax|}{|x|},\ |x| \le 1\right).$$

Matrix norms:

If the norm for the space is given by

$|x|_1 = \sum_{i=1}^{n} |x_i|$ then $||A||_1 = \max_p \sum_{i=1}^{n} |A_{i,p}|$ -- the maximum column sum.

If the norm for the space is given by

$|x|_2 = \sqrt{\sum_{p=1}^{n} |x_i|^2}$ then $||A||_2$ = the square root of the maximum eigenvalue of the matrix A (tranpose A).

If the norm for the space is given by

$|x|_\infty = \max_i |x_i|$ then $||A||_\infty = \max_p \sum_{i=1}^{n} |A_{p,i}|$ -- the maximum row sum.

Example: Consider the matrix

$$A = \begin{pmatrix} 4 & -2 & -1 & 0 \\ -2 & 4 & 0 & -1 \\ -1 & 0 & 4 & -2 \\ 0 & -1 & -2 & 4 \end{pmatrix}$$

The result presented in these notes establish that all eigenvalues lie in the disk $|\lambda - 4| \leq 3$. The improved result in the reference cited above show that all eigenvalues lie in the disks $|\lambda - 6| \leq 1$ and $|\lambda - 2| \leq 1$. Eigenvalues are actually 1, 3, 5, and 7.

Assignment

(4.1) Prove Theorem 9 for row sums instead of column sums.

(4.2) Let $$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Show that the eigenvalues λ of A satisfy $|\lambda-2| \leq 1$ or $\lambda = 1$. Give the eigenvalues for A. Get the Jordan form for A.