Section 5: Convergence

In many applications of mathematics, one can only make approximations. If you thought of making successive approximations, you would hope to be making a sequence which converges. The question is, in what sense would the approximation be good? In what sense would the sequence of approximation get close? This section sets a framework for understanding these questions by considering various forms of convergence.

Suppose that $\{E, <\ ,\ >\}$ is an inner product space of functions on an interval. We will discuss four methods of convergence: strong, weak, pointwise, and uniform.

Definition

A sequence of points $\{x_p\}$ converges to y *strongly* if $\lim_{p\to\infty} | x_p - y | = 0$.

A sequence of points $\{x_p\}$ converges to y *weakly* if, for each h in E,

$$\lim_{p\to\infty} < x_p - y, h> = 0.$$

A sequence of functions $\{f_p\}$ converges to g *pointwise on C* if for each x in C,

$$\lim_{p\to\infty} f_p(x) = g(x).$$

A sequence of functions $\{f_p\}$ converges to g *uniformly on C* if

$$\lim_{p\to\infty} \max | f_p(x) - g(x) | = 0.$$

Examples

(1) Strong convergence implies weak convergence, but not conversely.

Strong convergence   Weak convergence:

Suppose h is in E and $\{x_p\}$ converges to y *strongly* . By the Cauchy-Schwartz inequality $|< x_p - y, h>| \le |x_p - y|\ |h|$.

To see that weak convergence does not imply strong convergence, we need an example. One example comes from experiences with Fourier Series. Suppose h $L^2[0,1]$. There is a sequence a_p with $\lim_{p\to\infty} a_p = 0$ so that

$$h(x) = \sum_{p=0}^{\infty} a_p \sin(p\pi x).$$

Thus, the claim is that φ_p defined by $\varphi_p(x) = \sin(p\pi x)$, converges weakly to zero but not strongly. We have that for each h, $< \varphi_p - 0, h> = a_p \to 0$ and

$$| \varphi_p - 0 |^2 = \int_0^1 \sin^2(p\pi x)\, dx = \int_0^1 \frac{1+\cos(2p\pi x)}{2}\, dx = \frac{1}{2}.$$

(2) Uniform convergence implies pointwise convergence, but not conversely.

That uniform convergence implies point wise convergence is not so hard. To be assured that the implication does not go the other way, consider

$$f_n(x) = nxe^{-nx}$$

and note $f_n(1/n) = 1/e.$

Figure 5.1 is an illustrative graph. It shows this sequence converging pointwise, but not uniformly.

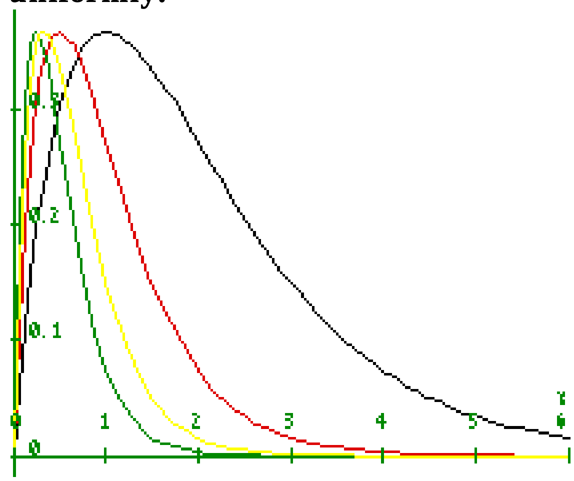


Figure 5.1

It's time to say what a Banach space is and what a Hilbert space is. Definitions: A sequence $\{x_p\}$ *converges* if

$$\lim_{m,n\to\infty} |\, x_m - x_n \,| \to 0.$$

A space is *complete* if every sequence that converges has a limit in the space. A *Banach space* is a complete, normed linear space. A *Hilbert space* is a complete, inner product space.

Another notion that is critical in the remainder of these notes and is likely in the vocabulary of the reader already is that of a *closed set*. A closed subset of a Hilbert space or Banach space is a set with the property that if $\{x_p\}$ is a sequence with values in the set and $\lim_p x_p = y$, then y is also in the set. There are alternate characterizations: closed sets are sets that contain all their boundary points, are sets whose complements are *open*.

Assignment:

(5.1) Show that if $|x_n| \to |L|$ and $x_n \to L$ weakly, then $x_n \to L$ strongly.
(Hint: Show that $|x_n - L|^2 \to |L|^2 - 2|L|^2 + |L|^2$.)

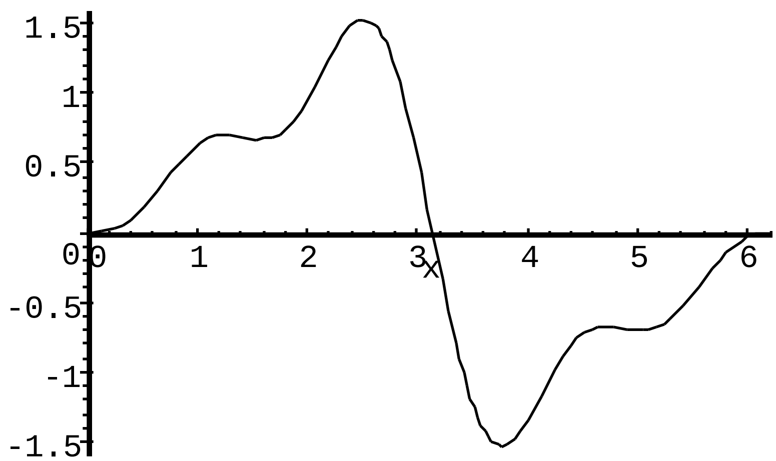
(5.2) Show that uniform convergence on [0,1] implies strong convergence in $L^2[0,1]$, but not conversely.

(5.3) Show that pointwise convergence on [0,1] does not imply strong convergence in $L^2[0,1]$ and that strong convergence in $L^2[0,1]$ does not imply pointwise convergence on [0,1].

(5.4) Discuss the nature of the convergence of

$$\sum_{1}^{\infty} (-1)^{n+1} \frac{\sin(nx)}{n}$$

on the interval $[0,2\pi]$. (Hint: here is the graph of $\sum_{1}^{4} (-1)^{n+1} \frac{\sin(nx)}{n}$.



(5.5) Suppose that $\lim_p x_p = y$ and $\lim_p u_p = v$.

(a) Show that $\lim_p < x_p, u_p> = < y, v>$.

(b) Show that $\lim_p | x_p | = | y |$.

(c) Let $\{\varphi_p\}$ be an infinite maximal orthonormal sequence, $x_p = \varphi_p$, and $u_p = \varphi_p/2$. Show that the weak limit of x_p and u_p is zero, but

$$\lim_p < x_p, u_p> \neq 0.$$

MAPLE Remark: Graphs of the first four functions in a sequence that converges pointwise, but not uniformly are made with MAPLE in a rather intuitive way.

```
> f1:=x->x*exp(-x); f2:=x->2*x*exp(-2*x); f3:=x->3*x*exp(-3*x);
  f4:=x->4*x*exp(-4*x);
> plot({f1(x),f2(x),f3(x),f4(x)},x=0..6);
```

You might prefer this syntax and graph.

```
> f:=(n,x)->n*x*exp(-n*x);
> plot3d(f(n,x),n=1..4,x=0..6,axes=NORMAL,orientation=[-10,50]);
```

The space C[-1,1] of continuous functions in L^2[-1,1] is an example of an innerproduct space that is not complete. When asked to show it is not complete, a novice might draw a sequence of functions in a manner suggested with this syntax:

```
>  plot({[x,1-2*x,x=0..1/2],[x,1-3*x,x=0..1/3],[x,1-4*x,x=0..1/4],
      [x,1+2*x,x=-1/2..0],[x,1+3*x,x=-1/3..0],[x,1+4*x,x=-1/4..0]},
      x=-1..1);
```

This is not a convincing picture for L^2[-1,1] is more complicated than suggested by that picture. To see this, establish that the sequence suggested by that picture converges to zero by seeing that

$$\int_{-1}^{1} [f_n(x) - 0]^2\, dx = 2 \int_{0}^{1/n} (1-nx)^2\, dx = 2/(3n).$$

```
> int((1-n*x)^2,x=0..1/n);
```

This integral goes to zero as n increases.

How does one make an example of a sequence in C[-1,1] that converges but has a limit not in C[-1,1]? Here's a suggestion.

```
> plot( {  [x,Pi*(1-x)/2,x=0..1], [x,-Pi*(1+x)/2,x=-1..0],
           [x,sum( sin(p*Pi*x)/p, p=1..5 ),x=-1..1]});
```

That graph suggests a good idea for an example. One must consider whether

$$\sum_{p} \sin(p\pi x)/p$$

converges in C([-1,1]). This brings up integrating the sum of the squares of the terms. Note the calculus here

```
> simplify(int( sin(p*Pi*x)*sin(q*Pi*x), x = -1..1));
```

This is zero if p and q are different integers. On the other hand

```
> subs(sin(p*Pi)=0, int( (sin(p*Pi*x)/p)^2 ,  x=-1..1));
```

We are ready to see the series is Cauchy. We see that the limit of the series is the odd extension of $\pi(1 - x)/2$.

```
> int((Pi*(1-x)/2)^2,x=0..1)
            - 2*sum(int(Pi*(1-x)*sin(p*Pi*x)/(2*p),x=0..1), p=1..n)
            + sum(int( (sin(p*Pi*x)/p )^2,x=0..1),p=1..n);

> limit(subs(sin(p*Pi)=0,"),n=infinity);
```

This limit is zero and we have that the sequence has limit not in C[-1,1].

Section 6: Orthogonality and Closest Point Projections

We now will study the notions of orthogonality and orthogonal projections. This will expand our notion of what a projection is. Critical in making up "closest point " projections is that the range of the projection should be convex. We examine this geometric notion in this section.

The identity of part 1 of the next theorem is called the *Pythagorean Identity* and the identity of part 2 is called the *Parallelogram Identity*.

Theorem 10 (1) If $<x, y> = 0$, then $|x+y|^2 = |x|^2 + |y|^2$.
(2) If x and y are in E, then $|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$.

What follows is a review of the notion of convexity for use in the exploration of projections.

Definition A set C is *convex* if for all x and y in C and all numbers t in [0,1], $tx + (1-t)y$ is in C.

REMARK: The next result gives a characterization of closest point projection in terms of the norm and also in terms of the dot product.

Theorem 11 If C is a closed, convex subset of the Hilbert space {E, < , >}, x_0 E, and $\delta = \inf\{ || x_0 - c ||, c \quad C\}$ then there is one and only one y_0 in C such that $|| x_0 - y_0 || = \delta$. Moreover, these are equivalent:
(a) z is in C and $\mathrm{Re}<x_0-z, c-z> \le 0$ for all c in C, and
(b) $z = y_0$.

<u>Outline of Proof:</u> Suppose that C is a closed convex set and that x_0 is in E. Let $\delta = \inf\{ |x_0-y| : y \quad C\}$

First we show there is a point y_0 in C such that $|x_0 - y_0| = \delta$. To do this define a sequence $\{u_p\}$ in C by $|u_p - x_0| < \delta + \frac{1}{p}$. Note that $\{u_p\}_{p=1}^{\infty}$ converges for $|u_n - u_m|^2 = |(u_n-x_0) - (u_m - x_0)|^2$
$= 2|u_n-x_0|^2 + 2|u_m-x_0|^2 - 4\ |(u_n+u_m)/2 - x_0|^2$ by the parallelogram identity. Because C is convex, $(u_n+u_m)/2$ is in C and this last line does not exceed $2|u_n-x_0|^2 + 2|u_m-x_0|^2 - 4\,\delta^2$ which goes to zero as n and m goes to infinity. Hence, the sequence $\{u_p\}$ converges strongly and has limit some point y_0 in E since E is complete. It is in C since C is closed.

Now we show there is only one closest point to x_0 in C. Suppose that

$$|x_0 - z| = \delta = |x_0 - y_0|.$$

Then

$$|y_0 - z|^2 = |(y_0 - x_0) - (z-x_0)|^2$$
$$= 2|y_0-x_0|^2 + 2|z-x_0|^2 - 4\ |(y_0+z)/2 - x_0|^2 = 0.$$

Thus, $y_0 = z$.

What follows is an understanding of the characterization of the closest point using the innerproduct.

These are equivalent:

(a) z is a point in C and Re$<x_0-z,c-z> \le 0$ for all c in C, and
(b) z is y_0.

a b Suppose that z has this property (a). We hope to show that $z = y_0$. Because of the uniqueness of this closest point, we would be content to show that $|x_0-z| = |x_0-y_0|$.

We know that $|x_0-y_0| \le |x_0-z|$.

$$0 \le |x_0-z|^2 - |x_0-y_0|^2 = |x_0-z|^2 - |(x_0-z)+(z-y_0)|^2$$
$$= |x_0-z|^2 - [|x_0-z|^2 - 2\,\mathrm{Re}<x_0-z,y_0-z> + |z-y_0|^2]$$
$$= 2\,\mathrm{Re}<x_0-z,y_0-z> - |z-y_0|^2 \le 0.$$

Therefore, $|x_0-z|^2 = |x_0-y_0|^2$.

Now we show that b a by showing that y_0 has this dot product characterization. Recall that the set C is convex so that if c is in C then so is $tc+(1-t)y_0$ in C. The first inequality holds because y_0 is the closest point in C to x_0:

$$0 \ge |x_0-y_0|^2 - |x_0-[tc+(1-t)y_0]|^2 = |x_0-y_0|^2 - |[x_0-y_0] + t(y_0 - c)|^2$$
$$= |x_0-y_0|^2 - |x_0-y_0|^2 + 2t\,\mathrm{Re}<x_0-y_0, c-y_0> - t^2\,|y_0 - c|^2 \text{ for all } 0 < t \le 1.$$

Therefore, Re$<x_0-y_0, c-y_0> \le 0$.

REMARK. This result leads to two irresistable questions:
(1) In a Banach Space that is not a Hilbert Space and given a point, is there a unique closest point on the unit disk? For example consider L_1 and L_∞?
(2) We have repeatedly used the parallelogram law. This holds in a Hilbert space. Does the parallelogram law hold in spaces other than Hilbert spaces?
(3) Convex sets, not necessarily closed, do not have to have points of minimum norm.
(4) Closed sets, not necessarily convex, do not have to have points of minimum norm. For example,

$$C = \;\;_{p=1}^{\infty} \{ e_p x \text{ where } 1+ 1/p \le x < \infty\}$$

is closed, not convex, and there is not a point of minimum norm.

Corollary 12 If M is a closed linear subspace of the Hilbert space {E, < , >}, x_0 is in E, and $\delta = \inf\{||x_0 - m||: m \;\; M\}$ then there is one and only one y_0 in M such that $||x_0 - y_0|| = \delta$. Moreover, $<x_0 - y_0 , m> = 0$ for all m in M. If $y_1 \;\; M$ and $y_1 \ne y_0$ then $<x_0 - y_1, m> \ne 0$ for all m in M.

Suggestion of Proof: Let x_0 be in E and y_0 be the closest point in M to x_0. There is such a point because M is closed and convex. We have from the above theorem that

$$\mathrm{Re}<x-y_0, m-y_0> \le 0 \text{ for all m in M.} \qquad (*)$$

We hope to show that

$$\mathrm{Re}\langle x-y_0, m-y_0\rangle = 0 \text{ for all } m \text{ in } M.$$

Let $u = 2y_0 - m$. Note that u is in M. Hence,

$$0 \geq \mathrm{Re}\langle x-y_0, u-y_0\rangle = \mathrm{Re}\langle x-y_0, y_0-m\rangle.$$

This last inequality, together with inequality (*) above gives that

$$\mathrm{Re}\langle x-y_0, y_0-m\rangle = 0 \text{ for all } m \text{ in } M.$$

We now want to show that this equality holds for the imaginary part, too.
Choose $v = y_0 + i(m-y_0)$. We have that

$0 = \mathrm{Re}\langle x-y_0, i(m-y_0)\rangle = \mathrm{Re}[-i\langle x-y_0, m-y_0\rangle$

$= \mathrm{Im}\langle x-y_0, m-y_0\rangle$.

Therefore, $\langle x-y_0, m-y_0\rangle = 0$ for m in the linear space.

Definition If C is a closed, convex set in E then P_c denotes the (possibly nonlinear) function from E to E such that if x is in E then $P_c(x)$ is the closest element in C to x.

Assignment

(6.1) Compute $\langle x,y\rangle$ and $|x+y|^2 - |x|^2 - |y|^2$ for $x = \{0,i\}$ and $y = \{0,1\}$.

(6.2) Suppose that $\{C_p\}_{p=1}^{n}$ is a collection of convex sets. Show that the intersection is also convex.

(6.3) Show that if $\{x_p\}_{p=1}^{n}$ is a sequence of points in a convex set C and

$$\sum_1^n t_p = 1, \text{ with each } t_p \geq 0,$$

then

$$\sum_1^n t_p x_p \text{ is in C.}$$

(6.4) The *closure* and the *interior* of a convex set is convex.

(6.5) If E N M and M and N are closed subspaces with $M \neq N$ then there is z in N, not in M and with z perpendicular to all points of M.
(Hint: Suppose that n is in N-M and m is in M. Let P(n) be the closest point to n that is in M and c = m + P(n). Then $0 = \langle n-P(n), c-P(n)\rangle$. Let z = n-P(n).)

(6.6) Consider the space R^2 with the norm given by $||x||_1 = |x_1| + |x_2|$.

(a) Show that every closed set in this space has a point of minimum norm.

(b) Show, by example, that this may fail to be unique.

(c) What happens if $||x||_\infty = \max(|x_1|, |x_2|)$?

(6.7) Note that $x = (x_1, x_2, \ldots)$ where $x_p = \frac{(-1)^p}{p}$ is in the real Hilbert space (little) RL^2 consisting of square summable real sequences and $C = \{y: y_p \geq 0, y \quad$ (little) $RL^2\}$ is a convex set in (little) RL^2. Find the closest element in C to x.

(6.8) Give an example of a convex set C such that if $P_c(x)$ is the closest element in C to x, then $P_c(x)$ is a function from E to E such that $P_c^2 = P_c$ but P_c is <u>not</u> necessarily linear. Can you characterize those convex sets in a Hilbert space for which the closest point projection is a linear function?

(6.9) Show that every closed, convex set in a Hilbert space has a point of minimum norm.

(6.10) Suppose that $\{\varphi_p\}_{p=1}^n$ is an orthonormal sequence and x is in E. Show that if $\{a_p\}_{p=1}^n$ is a number sequence then

$$|x - \sum_1^n a_p\varphi_p|^2 = |x|^2 + \sum_1^n |\langle x, \varphi_p\rangle - a_p|^2 - \sum_1^n |\langle x, \varphi_p\rangle|^2.$$

(Hint for seeing the above equality: expand the right side.)

How do you choose a_p such that $|x - \sum_1^n a_p\varphi_p|$ is minimum? Let C = span $\{\varphi_p\}_{p=1}^n$. Give a formula for $P_c(x)$.

(Remark: This problem has several interesting parts. There is the notion of a closest point,

that $\sum_1^n |\langle x, \varphi_p\rangle|^2$ converges as $n \to \infty$,

that $\sum_1^n \langle x, \varphi_p\rangle \varphi_p$ converges in E,

and that $x = \sum_1^n \langle x, \varphi_p\rangle \varphi_p$ if $\{\varphi_p\}$ is maximal orthonormal.)

(6.11) Show that $\varphi_n(x) = \sin(n\pi x)$, n = 1, 2, ... is an orthogonal sequence in $L^2[0,1]$. Let C be the span of $\varphi_n(x)$, n = 1,2,..., 100. Give a formula for $P_c(f)$ where f is given by $f(t) = t$ for $0 \leq t \leq 1$.

(6.12) Show that $\varphi_0(x) = 1$, $\varphi_1(x) = x$, $\varphi_2(x) = (3x^2 - 1)/2$, $\varphi_3(x) = (5x^3 - 3x)/2$, $\varphi_4(x) = (35x^4 - 30x^2 + 3)/8$ is an orthogonal sequence in $L^2[-1,1]$. Let C be the span of $\varphi_n(x)$, n = 0,1,...,4. Give a formula for $P_c(f)$ where $f(x) = x^3 + x^2 + x + 1$.

(6.13) Let

$$L_n(x) = \sum_{0}^{n} (-1)^p \binom{n}{p} \frac{1}{p!} x^p.$$

Compute L_0 through L_3 . Let

$$\Phi_n(x) = e^{-x} L_n(x).$$

Show that $\{\Phi_n(x)\}$ is an orthogonal sequence in $L^2([0,\infty), e^x)$. Let C be the span of Φ_n, n = 0,1,2,3. Give a formula for $P_c(x)$ where f(x) = .25 if $0 \le x \le 1$, = .75 if $2 \le x \le 3$, = 0 otherwise.

MAPLE Remark: When one thinks of the graph of cos(πx/2) on the interval [-1,1], one is struck by the resemblance of this graph to that of a quadratic, turned down and translated up, over the same interval. It's curious to think about how close one might approximate this transcandental function with a quadratic function.

The first thing to think of is the Taylor Polynomial of degree two.

```
> taylor(cos(Pi*x/2),x=0,3);
> plot({cos(Pi*x/2),1 - Pi^2*x^2/8},x=-1..1);
```

An alternate quadratic approximation that is more appropriate to this section is to use the polynomials of problem 6.12. These are called the Legendre polynomials. MAPLE knows these polynomials, and others. Here are the first three Legendre polynomials:

```
> with(orthopoly):
> P(0,x);P(1,x);P(2,x);
```

To get the best quadratic approximation for cos(πx/2) in $L^2[0,1]$ we compute the coefficients as in problem 6.10.

```
> a0:=int(cos(Pi*x/2)*P(0,x),x=-1..1)/int(P(0,x)^2,x=-1..1);
  a1:=int(cos(Pi*x/2)*P(1,x),x=-1..1)/int(P(1,x)^2,x=-1..1);
  a2:=int(cos(Pi*x/2)*P(2,x),x=-1..1)/int(P(2,x)^2,x=-1..1);
> plot({cos(Pi*x/2),a0*P(0,x) + a1*P(1,x) + a2*P(2,x)},x=-1..1);
```

To further emphasize the importance of orthogonal polynomials, we make one more illustration. Take n+1 points chosen evenly spaced on the interval [-1,1] and choose the point pairs {x[i],y[i]} where y[i] = f(x[i]). Then take the interpolating polynomial of degree n that fits the resulting n+1 point-pairs exactly. Do you think this polynomial sequence might converge uniformly to f? There is a classical example that illustrates how wrong this can be.

```
> f:=x->1/(1+25*x^2);
> for i from 1 to 9 do;
      s[i]:= -1 + (i-1)*2/8;
      fs[i]:= f(s[i]);
  od;
```

```
> fintrp:=x->interp([s[1],s[2],s[3],s[4],s[5],s[6],s[7],s[8],s[9]],
          [fs[1],fs[2],fs[3],fs[4],fs[5],fs[6],fs[7],fs[8],fs[9]],x);
> plot({f(x),fintrp(x)},x=-1..1);
```

The situation -- lack of closeness to f -- does not improve with more points; just modify this syntax to see.