Section 9: The Finite Dimension Paradigm

Since this collection of notes purports to be about linear operators on an inner product space, one would hope to develop an understanding of how to make examples. A model that is useful follows the form of the paradigm introduced in this section. Two useful applications of this paradigm are included. We return to finite dimensions.

Theorem 20 If A is a self adjoint, linear transformation from R^n to R^n then there is an orthonormal sequence $\{\varphi_p\}_{p=1}^{n}$ in R^n and a number sequence $\{\lambda_p\}_{p=1}^{n}$ such that if x is in R^n, then

$$Ax = \sum_{p=1}^{n} \lambda_p \langle x,\varphi_p\rangle \varphi_p.$$

<u>Suggestion of Proof:</u> Recall that corresponding to any A there is a sequence of projections such that $P_iP_j = 0$ if $i \neq j$ and such that $x = \Sigma\ P_i x$ for all x in E. Let $M_i = P_i(E)$. Note that if m M_i then $P_i m = m$ and $M_i \cap M_j = 0$ if $i \neq j$. Select in each M_i a maximal sequence of orthonormal vectors and name the mutually orthonormal vectors in the union of this collection for all i $\{\varphi_1, \varphi_2,\ldots,\varphi_k\}$. We argue that $k \leq n$, for R^n has at most n linearly independent vectors. Even more, it must be that k = n for otherwise, let y be perpendicular to their span. But, $y = \Sigma_i P_i y$, each $P_i y$ is in M_i, and the vectors in M_i cannot all be orthogonal to the the vectors in $\{\varphi_p\}$--otherwise, the sequence $\{\varphi_p\}$ was not chosen as directed. So, it must be that $\{\varphi_p\}$ forms an orthonormal basis for R^n. If we now assume that A is self adjoint, then each P_i is a closest point projection onto M_i. This means that P_i will have a representation in terms of the *Fourier coefficients*

$$u = \Sigma \langle u,\varphi_p\rangle \varphi_p \quad \text{and} \quad A = \Sigma\ \lambda_p\langle u,\varphi_p\rangle \varphi_p.$$

Corollary 21 Moreover, if m is a positive integer, then

$$A^m(x) = \sum_{p=1}^{n} \lambda_p{}^m \langle x, \varphi_p\rangle \varphi_p.$$

Remarks

(1) With A self adjoint,as in Theorem 20, we have that N(A)   = R(A).

(2) With A as supposed, if $\lambda_p \neq 0$ for $1 \leq p \leq k$ and $\lambda_p = 0$ for $k < p \leq n$, then u is in the null space of A if

$$u = \sum_{k+1}^{n} \langle u,\varphi_p\rangle \varphi_p$$

and v is in the range of A if

$$v = \sum_{1}^{k} \langle v,\varphi_p \rangle\, \varphi_p.$$

Definition If A is as in Theorem 20 and m is an integer such that $m \le n$ and $\lambda_i \ne 0$ for $1 \le i \le m$ and $\lambda_i = 0$ if $i > m$, then the *generalized inverse* is a linear function such that

$$A^\dagger(z) = \sum_{p=1}^{m} \frac{1}{\lambda_p} \langle z, \varphi_p \rangle\, \varphi_p$$

for each z in R^n.

Theorem 22 If A is as in Theorem 20, then $A^\dagger$ has the following properties:
(1) $AA^\dagger A = A$ (2) $A^\dagger AA^\dagger = A^\dagger$
(3) $AA^\dagger$ is the orthogonal projection onto R(A).
(4) $A^\dagger A$ is the orthogonal projection onto N(A) .

Remark:
(1) One uses the generalized inverse to get a "best solution" x for the equation Ax=y where A and y are given and where there is no x such that Ax=y. Suppose that y R^n and y   R(A). There is no x such that Ax = y. In order to find v in R(A) that is closest to y, choose $AA^\dagger y$. The above establishes $AA^\dagger$ as the closest point projection onto the range of A. Now, among all u's such that Au = v, the one with smallest norm is $A^\dagger y$. To see this, first note that $AA^\dagger y = v$. It remains to show that $A^\dagger y$ is the closest point to zero in the convex set of all points s that map to v, that is, we need to see that

$$\langle 0-A^\dagger y, s - A^\dagger y \rangle = 0$$

for all s such that As = v. To see this, note that $s - A^\dagger y$ is in N(A), and the above statement (4) shows that $A^\dagger y = A^\dagger A(A^\dagger y)$ is perpendicular to everything in the nullspace of A.

(2) If z is in R^n, then $AA^\dagger z$ is the closest point u to z in the range of A. Also, $A^\dagger(z)$ is the element with the smallest norm among all x's so that Ax = u.

Assignment

(9.1) Compute $A^\dagger$, $AA^\dagger$, and $A^\dagger A$ for $A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(9.2) Let $A = \begin{pmatrix} -3/2 & 1/2 & 0 \\ 1/2 & -3/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Verify that {1,1,1} is not in the range of A. Find v in the range of A so that $\| v - \{1,1,1\} \|$ is as small as possible. Find u so that Au = v and u has the smallest norm among all other x such that Ax = v.

(9.3) The following matrix is self-adjoint, but $A \ A^{+}$ is not diagonal:

$$\begin{pmatrix} 4/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 4/3 \end{pmatrix}$$

MAPLE Remark: The methods suggested in this section can be used with MAPLE to get the finite dimensional paradigm representation for a matrix.

```
> with(linalg):
> A:= array([[-3/2, 1/2, 0], [1/2, -3/2, 0],[0, 0, 0]]);
> eigenvects(A);
> v1:=vector([-1,1,0]); v2:=vector([1,1,0]); v3:=vector([0,0,1]);
> u:=vector([x,y,z]);
> rep:= (-2)*innerprod(u,v1)*v1/norm(v1,2)^2 +
             (-1)* innerprod(u,v2)*v2/norm(v2,2)^2 +
                   (0)*innerprod(u,v3)*v3/norm(v3,2)^2;
```

To understand the next MAPLE command, note that if u and v are vectors and α and β are numbers, then

add(u,v,α,β)

computes α **u** + β **v.**

```
> add(v1,v2,-x+y,-(x+y)/2);
```

As a check, it should be noted that the results of the last line are the same as if regular matrix multiplication had been performed. The value of this representation has been explained in this section.

Section 10: Bounded, linear maps from E to ℂ

The notion of continuity is basic to any analysis of functions. In a study of Hilbert spaces, one should have in their repertoire examples of linear functions that are continuous and examples that are not continuous. We look first at the simplest examples of continuous linear functions on an inner product space: functions from E to the complex plane.

We provide a characterization of bounded, linear functions from a Hilbert space to the complex plane.

Theorem 23 Suppose that L is a linear function from {E,< , >} to E. These are equivalent:

(a) There is a number b such that if x is in E then $|L(x)| \leq b\ |x|$ - - L is *bounded*.

(b) if $\{x_p\}_{p=1}^{\infty}$ is a sequence in E with limit y then $\lim_p L(x_p) = L(y)$ - - L is *continuous*.

<u>Suggestion of Proof:</u>

a b $|L(x_p) - L(y)| = |L(x_p-y)| \leq b\ |x_p-y|$.

b a Suppose L is continuous and there is no such b. For each p, there is z_p with $|L(z_p)| \geq p\ |z_p|$. Let $x_p = \frac{z_p}{|z_p|\sqrt{p}}$. Then $x_p \to 0$, but

$$|L(x_p)| = \frac{|L(z_p)|}{|z_p|\sqrt{p}} \geq \sqrt{p},$$

so that $|L(x_p)| \to \infty$. This contradicts the assumption that L is continuous.

Remark: The smallest b is the *norm* of A. (Recall the earlier discussion on the norm of a matrix in Chapter 4.)

Theorem 24 Suppose that $L \neq 0$ is a bounded linear function and M = N(L). Then M is a closed, linear subspace. Moreover, if L: E → ℂ then dim M = 1.

<u>Suggestion of Proof:</u> First note that M is a closed linear subspace. Suppose that L maps E to C. To prove that dim(M) = 1, suppose that it is not. Let x and y be linearly independent members of M · and consider

$$z = \frac{x}{L(x)} - \frac{y}{L(y)}.$$

Since M is a subspace, z is in M and L(z) = 0. Thus z M∩M and hence must be 0. Hence

$$x = \frac{L(x)}{L(y)}\, y$$

and x and y are not linearly independent.

Theorem 25 (***The Riesz Representation Theorem***)
Suppose that A is a linear function from {E, < , >} to $\mathbb{C}$. These are equivalent:
(a) There is y in E such that A(x) = <x, y> for all x in E, and
(b) A is bounded.

<u>Suggestion of Proof:</u>
a b $|Ax| = |\langle x,y\rangle| \leq |x|\ |y|$.
b a By 24, N(A) has dimension 1. Let z be in N(A) with |z| = 1. Let

$$y = A(z)^* z.$$

Then,
$\langle x,y\rangle = \langle x, A(z)^* z\rangle = A(z) \langle x,z\rangle$
= A(z) {<Px, z> + <(1-P)x, z>} where P is the closest point projection onto N(A)
$= A(z) \langle (1-P)x,z\rangle = A(z) \langle \beta z,z\rangle$ **since N(A) is one dimensional**
$= \beta\, A(z) = A(\beta z) = A((1-P)x) = A(x)$

Assignment
(10.1) Let L be a function on L^2 defined by

$$L(x) = \sum_{p=1}^{\infty} |\langle x,e_p\rangle|^2.$$

Is L a linear function from L^2 to $\mathbb{C}$?

(10.2) Let L be defined on R^3 by $L(x) = x_1 + 3x_2 - 7x_3$.
(a) Find v such that L(x) = <x,v> for all x in R^3.
(b) Find b such that $|L(x)| \leq b\ |x|$.

(10.3) Let $A:L^2[-1,1] \to \mathbb{C}$ by

$$A(f) = \int_0^{1/2} f(x)\, dx.$$

Show that A is continuous and find g in $L^2[-1,1]$ such that A(f) = <f, g>.

(10.4) Let C[-1,1] be the continuous functions in $L^2[-1,1]$ and A: C[-1,1] $\to \mathbb{C}$ by A(f) = f(1/2). Show that A is not continuous, that is, show there is a sequence f_p in C[-1,1] with $\lim_{p\to\infty} f_p = 0$, but $\lim_p A(f_p) \neq A(0)$. Show that there is no g in $L^2[-1,1]$ such that

$$A(f) = \int_{-1}^{1} f(x)\, g(x)\, dx.$$

(Hint: Recall the sequence of functions $\{f_n\}$ that converges strongly, but not pointwise.)

(10.5) For each f in $L^2[0,1]$, let Y(t) be a function such that y′ + 3 y = f, with y(0) = 0. Define $A:L^2[0,1] \to R$ by

$$A(f) = \int_0^1 y(t)\, dt.$$

Show that A is a bounded linear function on $L^2[0,1]$ and find g such that A(f) = < f, g >.

MAPLE Remark: This is an exercise in getting the Riesz representation of a continuous mapping from $L^2[0,1]$ to the complex plane.

Pick x in [0,1]. We define a point - -a function -- G(x) in $L^2[0,1]$. Since points in $L^2[0,1]$ are functions on [0,1], we write this G as a function of not only x, but also t: G(x,t). The linear mapping as defined by the Riesz theorem will be

$$< G(x,t), f(t) > = \int_0^1 G(x,t)\, f(t)\, dt.$$

It remains to say what is the linear mapping for which we will get this Riesz representation. Here is the definition of L:

For each f, we define L(f) = y where

$$y'' + 3\, y' + 2y = f, \text{ with } y(0) = y(1) = 0.$$

We use MAPLE to create this y. Then we make G(x,).

First we solve the differential equation with y(0) = 0. This will leave one constant to be evaluated.

```
> dsolve({diff(y(x),x,x)+3*diff(y(x),x)+2*y(x)=f(x),y(0)=0},y(x));
> _C2:=int( (exp(-1+u)-exp(-2*(1-u)))*f(u), u=0..1)/(exp(-1)-exp(-2));
> ("");
```

The above makes G(x,u). Input f, evaluate the integrals, and one has a number. The Riesz theorem says that we can find a representation in term of the dot product, which is an integral in this case. With a study of those integrals, we can extract G:

```
> G:=proc(x,u)
        if u < x then
               (exp(-x+u)-exp(-2*(x-u)))
               -(exp(-1+u)-exp(-2+2*u))*exp(-x)/(exp(-1)-exp(-2))
               +(exp(-1+u)-exp(-2+2*u))*exp(-2*x)/(exp(-1)-exp(-2))
         else -(exp(-1+u)-exp(-2+2*u))*exp(-x)/(exp(-1)-exp(-2))
               +(exp(-1+u)-exp(-2+2*u))*exp(-2*x)/(exp(-1)-exp(-2))
        fi
        end;
```

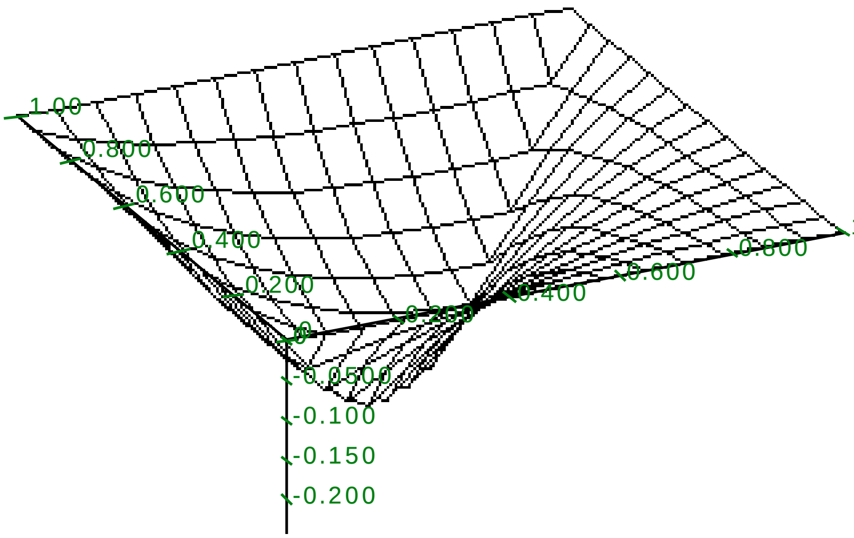
There are some properties of Green's functions that may be familiar. It satisfies the differential equation for x < u and for x > u. We verify this:

```
> z1:=x->(exp(-x+u)-exp(-2*(x-u)))
           -(exp(-1+u)-exp(-2+2*u))*exp(-x)/(exp(-1)-exp(-2))
           +(exp(-1+u)-exp(-2+2*u))*exp(-2*x)/(exp(-1)-exp(-2));

> diff(z1(x),x,x)+3*diff(z1(x),x)+2*z1(x);
> z2:=x->-(exp(-1+u)-exp(-2+2*u))*exp(-x)/(exp(-1)-exp(-2))
           +(exp(-1+u)-exp(-2+2*u))*exp(-2*x)/(exp(-1)-exp(-2));
> diff(z2(x),x,x)+3*diff(z2(x),x)+2*z2(x);
```

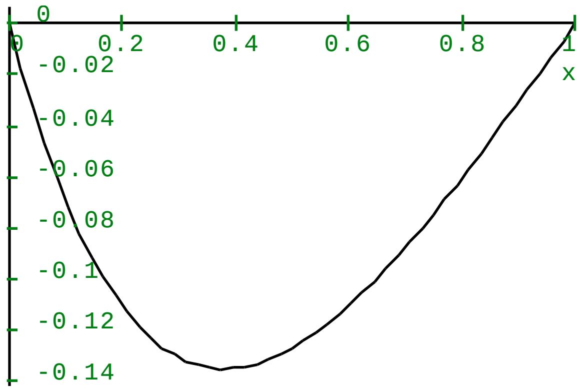
There is a symmetry for the graph of G:

```
> plot3d(G,0..1,0..1);
```



The observation from the graph that the values of G are negative gives information about the solution for f's that have only non-negative values. We compute a solution for the boundary value problem with f the constant function 1.

```
> w:=x->int(-(exp(-1+u)-exp(-2+2*u))*exp(-x)/(exp(-1)-exp(-2))
      +(exp(-1+u)-exp(-2+2*u))*exp(-2*x)/(exp(-1)-exp(-2)),u=x..1)
      + int((exp(-x+u)-exp(-2*(x-u)))
      -(exp(-1+u)-exp(-2+2*u))*exp(-x)/(exp(-1)-exp(-2))
      +(exp(-1+u)-exp(-2+2*u))*exp(-2*x)/(exp(-1)-exp(-2)),
           u=0..x);
> w(0);w(1);diff(w(x),x,x)+3*diff(w(x),x)+2*w(x)-1;
> simplify(");
> simplify(");
> plot(w(x),x=0..1);
```


0
0
0.2
0.4
0.6
0.8
1
x
-0.02
-0.04
-0.06
-0.08
-0.1
-0.12
-0.14