Module 4: The Gramm Schmidt Process

Having a set of orthogonal vectors
\[ \{ \Theta_1, \Theta_2, \Theta_3 ... \} \]
has become an important idea. These may be converted into an orthonormal set by dividing by the norm:

\[ \Phi_n = \frac{\Theta_n}{\| \Theta_n \|} \]
From the Fourier Inequality, if \( f \) is in the space then the element of the span of the \( \Phi_n \)'s that is closest to \( f \) is
\[
\sum_p \langle f, \Phi_p \rangle \Phi_p.
\]
This last result is why Fourier Series is a topic for study by scientists and engineers.
Suppose that \( \{ v_1, v_2, v_3, \ldots \} \) are linearly independent, but not orthogonal. We create an orthonormal sequence that spans the same set. This is the

**GRAMM SCHMIDT PROCESS.**
Procedure: The GRAMM SCHMIDT PROCESS

Step 1:

\[ \Theta_1 = v_1 \]
\[ \Phi_1 = \frac{\Theta_1}{\|\Theta_1\|} \]

Note that \( \|\Phi_1\| = 1 \).

How do we make the next term?
Step 2:

\[ \Theta_2 = v_2 - \langle v_2, \Phi_1 \rangle \Phi_1 \]
\[ \Phi_2 = \Theta_2 / \| \Theta_2 \| \]

Note that

a. \( \langle \Phi_1, \Phi_2 \rangle = 0 \),

b. \( \| \Phi_2 \| = 1 \).

How is the third one made?
Step 3:

$$\Theta_3 = v_3 - \langle v_3, \Phi_2 \rangle \Phi_2 - \langle v_3, \Phi_1 \rangle \Phi_1,$$
$$\Phi_3 = \Theta_3 / \| \Theta_3 \|.$$  

Note that

a. $$\langle \Phi_3, \Phi_1 \rangle = 0 = \langle \Phi_3, \Phi_2 \rangle$$

b. $$\| \Phi_3 \| = 1.$$  

The process continues.
Example in $\mathbb{R}^3$.
We perform the Gramm Schmidt Process to the two vectors $v_1 = [1, 1, 1]$ and $v_2 = [1, 0, 1]$.

$\Theta_1 = [1, 1, 1]$, $\Phi_1 = [1, 1, 1]/\sqrt{3}$

$\Theta_2 = [1, -2, 1]/\sqrt{3}$, $\Phi_2 = [1, -2, 1]/\sqrt{6}$.

Check that these are correct. Give the closest point in the plane spanned by $[1, 1, 1]$ and $[1, 0, 1]$ to $[1, 2, 3]$. 

This process can be done to functions, too. We perform the Gramm Schmidt Process in $\mathbb{C}(\mathbb{[-1,1]})$ to the functions $1, x, x^2, \text{ and } x^3$.

**Step 1:**

\[ \Theta_1 = v_1 \quad \Phi_1 = \Theta_1 / \| \Theta_1 \| \]

**Step 2:**

\[ \Theta_2 = v_2 - \langle v_2, \Phi_1 \rangle \Phi_1 \]
\[ \Phi_2 = \Theta_2 / \| \Theta_2 \| \]

**Step 3:**

\[ \Theta_3 = v_3 - \langle v_3, \Phi_2 \rangle \Phi_2 - \langle v_3, \Phi_1 \rangle \Phi_1 \]
\[ \Phi_3 = \Theta_3 / \| \Theta_3 \| \].
Assignment: See the Maple Worksheet

In this module 4, we have
1. explained the Gramm Schmidt Process,
2. worked out a 3 dimensional example, and
3. referenced a Maple worksheet to work an example in $C([-1,1])$. 